CHAPTER 1

Sets

SOME IMPORTANT NUMBER SETS

N = Set of all natural numbers

W = Set of all whole numbers

Z or I = set of all integers

Z^+ = Set of all +ve integers

Z^- = Set of all –ve integers

Z_0 = The set of all non-zero integers.

Q = The set of all rational numbers.

R = The set of all real numbers.

$R - Q$ = The set of all irrational numbers

Some Operation on Sets

(*i*) **De-Morgan Laws:** $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$

(*ii*) $A - (B \cup C) = (A - B) \cap (A - C)$; $A - (B \cap C) = (A - B) \cup (A - C)$

(*iii*) **Distributive Laws:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(*iv*) **Commutative Laws:** $A \cup B = B \cup A$; $A \cap B = B \cap A$

(*v*) **Associative Laws:** $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$

(*vi*) $A \cap \phi = \phi$; $A \cap U = A$

$A \cup \phi = A$; $A \cup U = U$

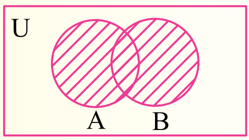
(*vii*) $A \cap B \subseteq A$; $A \cap B \subseteq B$

(viii) $A \subseteq A \cup B$; $B \subseteq A \cup B$

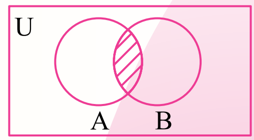
(ix) $A \subseteq B \Rightarrow A \cap B = A$

(x) $A \subseteq B \Rightarrow A \cup B = B$

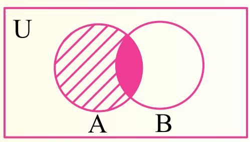
VENN Diagram



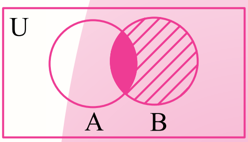
$A \cup B$



$A \cap B$

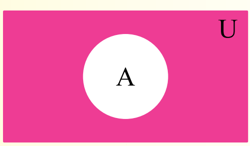


$A - B$

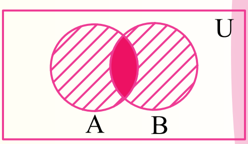


$B - A$

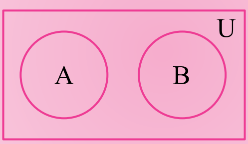
Clearly $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$



A'



$(A \Delta B) = (A - B) \cup (B - A)$



Disjoint Sets

Note: $A \cap A' = \phi$, $A \cup A' = U$

Some Important Results on Number of Elements in Sets

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.

(iii) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(A \Delta B)$ = No. of elements which belong to exactly one of A or B

$= n((A - B) \cup (B - A))$

$= n(A - B) + n(B - A)$ $\quad [\because (A - B)$ and $(B - A)$ are disjoint$]$

$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

(*v*) $n(A \cup B \cup C)$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - (A \cap C) + n(A \cap B \cap C)$

(*vi*) Number of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(*vii*) Number of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(*viii*) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(*ix*) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

(*x*) If A_1, A_2 .... A_n are finite sets, then

$$n\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} n(A_i) - \sum_{1 \le i < j \le n} n(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le n} n(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} n(A_1 \cap A_2 \cap \ldots A_n)$$

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CHAPTER 2

Relations and Functions

RELATION

If A and B are two non-empty sets, then a relation R and from A to B is a subset of $A \times B$.

If $R \subseteq A \times B$ and $(a, b) \in R$, then we say that a is related to b by the relation R, written as aRb.

If $R \subseteq A \times A$, then we simply say R is a relation on A.

REPRESENTATION OF A RELATION

(*i*) **Roster form:** In this form, we represent the relation by the set of all ordered pairs belongs to R.

(*ii*) **Set-builder form:** In this form, we represent the relation R from set A to set B as

$R = \{(a, b) : a \in A, b \in B$ and the rule which relate the elements of A and $B\}$.

DOMAIN, CODOMAIN AND RANGE OF A RELATION

Let R be a relation from a non-empty set A to a non-empty set B. Then, set of all first components or coordinates of the ordered pairs belonging to R is called the **domain** of R, while the set of all second components or coordinates of the ordered pairs belonging to R is called the **range** of R. Also, the set B is called the **codomain** of relation R.

Thus, domain of $R = \{a : (a, b) \in R\}$ and range of $R = \{b : (a, b) \in R\}$

TYPES OF RELATIONS

(*i*) **Empty or Void Relation:** As $\phi \subset A \times A$, for any set A, so ϕ is a relation on A, called the empty or void relation.

(*ii*) **Universal Relation:** Since, $A \times A \subseteq A \times A$, so $A \times A$ is a relation on A, called the universal relation.

(*iii*) **Identity Relation:** The relation $I_A = \{(a, a): a \in A\}$ is called the identity relation on A.

(*iv*) **Reflexive Relation:** A relation R on a set A is said to be reflexive relation, if every element of A is related to itself.

Thus, $(a, a) \in R, \forall a \in A \Rightarrow R$ is reflexive.

(*v*) **Symmetric Relation:** A relation R on a set A is said to be symmetric relation iff $(a,b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$

i.e. $a R b \Rightarrow bRa, \forall a, b \in A$

(*vi*) **Transitive Relation:** A relation R on a set A is said to be transitive relation, iff $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $(a, c) \in R, \forall a, b, c \in A$

EQUIVALENCE RELATION

A relation R on a set A is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on A.

IMPORTANT RESULTS ON RELATION

(*i*) If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

(*ii*) The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

(*iii*) If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.

(*vi*) Let A and B be two non-empty finite sets consisting of m and n elements, respectively. Then, $A \times B$ consists of mn ordered pairs. So, the total number of relations from A to B is 2^{nm}.

(*v*) If a set A has n elements, then number of reflexive relations from A to A is 2^{n^2-n}.

FUNCTIONS

Let A and B be two non-empty sets, then a function f from set A to set B is a rule which associates each element of A to a unique element of B.

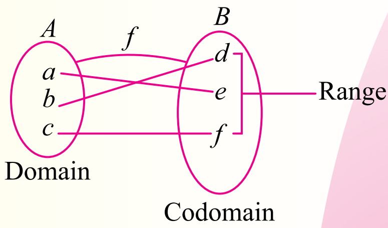
DOMAIN, CODOMAIN AND RANGE OF A FUNCTION

If f : $A \to B$ is a function from A to B, then

(*i*) the set A is called the domain of $f(x)$.

(*ii*) the set B is called the codomain of $f(x)$.

(*iii*) the subset of B containing only the images of elements of A is called the **range** of $f(x)$.



NUMBER OF FUNCTIONS

Let X and Y be two finite sets having m and n elements repectively. Then each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is n^m.

NUMBER OF ONE-ONE FUNCTIONS

Let A and B are finite sets having m and n elements repectively, then the number of one-one functions from A to B is $\begin{cases} {}^nP_m, & n \geq m \\ 0, & n < m \end{cases}$

$$= \begin{cases} n(n-1)(n-2)\ldots(n-(m-1)), & n \geq m \\ 0, & n < m \end{cases}$$

NUMBER OF ONTO (OR SURJECTIVE) FUNCTIONS

Let A and B are finite sets having m and n elements respectively, then number of onto (or surjective) functions from A to B is

$$= \begin{cases} n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \ldots, & n < m \\ n!, & n = m \\ 0, & n > m \end{cases}$$

NUMBER OF BIJECTIVE FUNCTIONS

Let A and B are finite sets having m and n elements respectively, them number of bijective functions from A to B is $\begin{cases} n!, \text{ if } n = m \\ 0, \text{ if } n > m \text{ or } n < m \end{cases}$

Properties of modulus function

For any $x, y, a \in R$.

(i) $|x| \geq 0$

(ii) $|x| = |-x|$

(iii) $|xy| = |x||y|$

(iv) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}; y \neq 0$

(v) $|x| = a \Rightarrow x = \pm a$

(vi) $\sqrt{x^2} = |x|$

(vii) $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$. where a is positive.

(viii) $|x| \leq a \Rightarrow x \in [-a, a]$. where a is positive

(ix) $|x| > |y| \Rightarrow x^2 > y^2$

(x) $\left||x| - |y|\right| \leq |x| + |y| = \begin{cases} (a)\, |x| + |y| = |x + y| \Rightarrow xy \geq 0 \\ (b)\, |x| + |y| = |x - y| \Rightarrow xy \leq 0 \end{cases}$

PROPERTIES OF GREATEST INTEGER FUNCTION

(i) $[x + n] = n + [x], n \in I$

(ii) $[-x] = -[x], x \in I$

(iii) $[-x] = -[x] - 1, x \notin I$

(iv) $[x] \geq n \Rightarrow x \geq n, n \in I$

(v) $[x] > n \Rightarrow x \geq n + 1, n \in I$

(vi) $[x] \leq n \Rightarrow x < n + 1, n \in I$

(vii) $[x] < n \Rightarrow x < n, n \in I$

(viii) $[x + y] = [x] + [y + x - [x]]$ for all $x, y \in R$

(ix) $[x + y] \geq [x] + [y]$

(x) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx], n \in N.$

IMPORTANT POINTS TO BE REMEMBERED

(i) Constant function is periodic with no fundamental period.

(ii) If $f(x)$ is periodic with period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ are also periodic with same period T.

(iii) If $f(x)$ is periodic with period T_1 and $g(x)$ is periodic with period T_2. then $f(x) + g(x)$ is periodic with period equal to

(a) LCM of $\{T_1, T_2\}$, if there is no positive k, such that $f(k + x) = g(x)$ and $g(k + x) = f(x)$.

(b) $\frac{1}{2}$LCM of $\{T_1, T_2\}$, if there exist a positive number k such that $f(k + x) = g(x)$ and $g(k + x) = f(x)$

(iv) If $f(x)$ is periodic with period T, then $kf(ax + b)$ is periodic with period $\frac{T}{|a|}$, where $a, b, k \in R$ and $a, k \neq 0$.

(v) If $f(x)$ is a periodic function with period T and $g(x)$ is any function, such that range of $f \subseteq$ domain of g, then gof is also periodic with period T.

Properties of Even and Odd Functions

(i) Even function ± Even function = Even function.

(ii) Odd function ± Odd function = Odd function.

(iii) Even function × Odd function = Odd function.

(iv) Even function × Even function = Even function.

(v) Odd function × Odd function = Even function.

(vi) gof or fog is even, if both f and g are even or if f is odd and g is even or if f is even and g is odd.

(vii) gof or fog is odd, if both of f and g are odd.

(viii) If $f(x)$ is an even function, then $\frac{d}{dx}f(x)$ is an odd function and if $f(x)$ is an odd function, then $\frac{d}{dx}f(x)$ is an even function.

(ix) The graph of an even function is symmetrical about Y-axis.

(x) The graph of an odd function is symmetrical about origin or symmetrical in opposite quadrants.

(xi) An even function can never be one-one, however an odd function may or may not be one-one.

Properties of inverse function

(a) The inverse of a bijection is unique.

(b) If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively. If $fof = I$, then f is inverse of itself.

(c) The inverse of a bijection is also a bijection.

(d) If f & g are two bijections $f: A \to B$, $g : B \to C$ & gof exist, then the inverse of gof also exists and $(gof)^{-1} = f^{-1} og^{-1}$.

(e) Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of 'f' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$.

The graph of f^{-1} obtained by reflecting the graph of f about the line $y = x$.

GENERAL

If x, y are independent variables, then :

(a) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\ell nx$

(b) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n$, $n \in R$ or $f(x) = 0$

(c) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$ or $f(x) = 0$

(d) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

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CHAPTER 3

Trigonometric Ratio and Identities

RELATION BETWEEN SYSTEM OF MEASUREMENT OF ANGLES

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}$$

$$1 \text{ Radian} = \frac{180}{\pi} \text{ degree} \approx 57^\circ\ 17'\ 15'' \text{ (approximately)}$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radian} \approx 0.0175 \text{ radian}$$

BASIC TRIGONOMETRIC IDENTITIES

(*a*) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(*b*) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(*c*) If $\sec\theta + \tan\theta = k \Rightarrow \sec\theta - \tan\theta = \frac{1}{k} \Rightarrow 2\sec\theta = k + \frac{1}{k}$

(*d*) $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$

(*e*) If $\text{cosec}\,\theta + \cot\theta = k \Rightarrow \text{cosec}\,\theta - \cot\theta = \frac{1}{k} \Rightarrow 2\,\text{cosec}\,\theta = k + \frac{1}{k}$

TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES

(*a*) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$.

(*b*) $\sin(-\theta) = -\sin\theta$ $\quad$ $\cos(-\theta) = \cos\theta$

$\sin(90^\circ - \theta) = \cos\theta$ $\quad$ $\cos(90^\circ - \theta) = \sin\theta$

$\sin(90° + \theta) = \cos\theta$ $\qquad$ $\cos(90° + \theta) = -\sin\theta$

$\sin(180° - \theta) = \sin\theta$ $\qquad$ $\cos(180° - \theta) = -\cos\theta$

$\sin(180° + \theta) = -\sin\theta$ $\qquad$ $\cos(180° + \theta) = -\cos\theta$

$\sin(270° - \theta) = -\cos\theta$ $\qquad$ $\cos(270° - \theta) = -\sin\theta$

$\sin(270° + \theta) = -\cos\theta$ $\qquad$ $\cos(270° + \theta) = \sin\theta$

Note:

(i) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$.

(ii) $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n$; $\cos(2n+1)\dfrac{\pi}{2} = 0$ where $n \in I$.

IMPORTANT TRIGONOMETRIC FORMULA

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

(v) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A + B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A + B) = \dfrac{\cot B \cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A - B) = \dfrac{\cot B \cot A + 1}{\cot B - \cot A}$

(ix) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$.

(x) $2\cos A \sin B = \sin(A + B) - \sin(A - B)$.

(xi) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$.

(xii) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$.

(xiii) $\sin C + \sin D = 2\sin\left(\dfrac{C + D}{2}\right)\cos\left(\dfrac{C - D}{2}\right)$

(*xiv*) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(*xv*) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(*xvi*) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

(*xvii*) $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(*xviii*) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

(*xix*) $1 + \cos 2\theta = 2\cos^2\theta$ or $\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$

(*xx*) $1 - \cos 2\theta = 2\sin^2\theta$ or $\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$

(*xxi*) $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta}{1+\cos 2\theta} = \pm\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$

(*xxii*) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

(*xxiii*) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(*xxiv*) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(*xxv*) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(*xxvi*) $\sin^2 A - \sin^2 B = \sin(A+B)\cdot\sin(A-B) = \cos^2 B - \cos^2 A$.

(*xxvii*) $\cos^2 A - \sin^2 B = \cos(A+B)\cdot\cos(A-B)$.

(*xxviii*) $\sin(A+B+C)$

$= \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B$

$- \sin A \sin \mathrm{B} \sin \mathrm{C}$

$= \Sigma \sin A\cos B\cos C - \Pi \sin A$

$= \cos A\cos B\cos C\,[\tan A + \tan B + \tan C - \tan A\tan B\tan C]$

(xxix) $\cos(A+B+C)$

$= \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C$

$- \cos A \sin B \sin C$

$= \Pi \cos A - \Sigma \sin A \sin B \cos C$

$= \cos A \cos B \cos C\,[1 - \tan A \tan B - \tan B \tan C - \tan C \tan A]$

(xxx) $\tan(A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{S_1 - S_3}{1 - S_2}$$

(xxxi) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin\,(\alpha + \overline{n-1}\beta)$

$$= \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

(xxxii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\,(\alpha + \overline{n-1}\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

VALUES OF SOME T-RATIOS FOR ANGLES 18°, 36°, 15°, 22.5°, 67.5° ETC.

(a) $\sin 18° = \frac{\sqrt{5}-1}{4} = \cos 72° = \sin\frac{\pi}{10}$

(b) $\cos 36° = \frac{\sqrt{5}+1}{4} = \sin 54° = \cos\frac{\pi}{5}$

(c) $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75° = \sin\frac{\pi}{12}$

(d) $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \cos\frac{\pi}{12}$

(e) $\tan\frac{\pi}{12} = 2 - \sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot\frac{5\pi}{12}$

(*f*) $\tan\frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot\frac{\pi}{12}$

(*g*) $\tan(22.5°) = \sqrt{2}-1 = \cot(67.5°) = \cot\frac{3\pi}{8} = \tan\frac{\pi}{8}$

(*h*) $\tan(67.5°) = \sqrt{2}+1 = \cot(22.5°)$

MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS

(*a*) $a\cos\theta + b\sin\theta$ will always lie in the interval $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(*b*) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$, where $a, b > 0$.

(*c*) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta)$

$\le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

(*d*) Minimum value of $a^2\cos^2\theta + b^2\sec^2\theta$ is either $2ab$ or a^2+b^2, if for some real θ equation $a\cos\theta = b\sec\theta$ is true or not true $\{a, b > 0\}$.

(*e*) Minimum value of $a^2\sin^2\theta + b^2\text{cosec}^2\theta$ is either $2ab$ or a^2+b^2, if for some real θ equation $a\sin\theta = b\,\text{cosec}\,\theta$ is true or not true $\{a, b > 0\}$.

IMPORTANT RESULTS

(*a*) $\sin\theta\sin(60°-\theta)\sin(60°+\theta) = \frac{1}{4}\sin 3\theta$

(*b*) $\cos\theta\cos(60°-\theta)\cos(60°+\theta) = \frac{1}{4}\cos 3\theta$

(*c*) $\tan\theta\tan(60°-\theta)\tan(60°+\theta) = \tan 3\theta$

(*d*) $\cot\theta\cot(60°-\theta)\cot(60°+\theta) = \cot 3\theta$

(*e*) (*i*) $\sin^2\theta + \sin^2(60°+\theta) + \sin^2(60°-\theta) = \frac{3}{2}$

(*ii*) $\cos^2\theta + \cos^2(60°+\theta) + \cos^2(60°-\theta) = \frac{3}{2}$

(*e*) (*i*) If $\tan A + \tan B + \tan C = \tan A\tan B\tan C$,

then $A + B + C = n\pi$, $n \in I$.

(*ii*) If $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$,

then $A + B + C = (2n+1)\frac{\pi}{2}$, $n \in I$.

(g) $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos(2^{n-1}\theta) = \dfrac{\sin(2^n \theta)}{2^n \sin\theta}$

(h) $\cot A - \tan A = 2 \cot 2A$.

CONDITIONAL IDENTITIES

If $A + B + C = 180°$, then

(a) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(b) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(c) $\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} = 1$

(d) $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$

(e) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(f) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(g) $\sin A + \sin B + \sin C = 4 \cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(h) $\cos A + \cos B + \cos C = 1 + 4 \sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

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CHAPTER 4

Trigonometric Equations

GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS (TO BE REMEMBERED)

(*a*) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers).

(*b*) If $\cos\theta = 0$, then $\theta = (2n+1)\dfrac{\pi}{2}$ $n \in I$.

(*c*) If $\tan\theta = 0$, then $\theta = n\pi$, $n \in I$.

(*d*) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$, $n \in I$.

(*e*) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0, \pi]$.

(*f*) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$.

(*g*) If $\sin\theta = 1$, then $\theta = 2n\pi + \dfrac{\pi}{2} = (4n+1)\dfrac{\pi}{2}$, $n \in I$.

(*h*) If $\cos\theta = 1$ then $\theta = 2n\pi$, $n \in I$.

(*i*) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$, then $\theta = n\pi \pm \alpha$, $n \in I$.

(*j*) For $n \in I$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, $n \in I$

$\sin(n\pi + \theta) = (-1)^n \sin\theta$

$\cos(n\pi + \theta) = (-1)^n \cos\theta$

(*k*) $\cos n\pi = (-1)^n$, $n \in I$.

(*l*) If n is an odd integer then $\sin \frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$, $\cos \frac{n\pi}{2} = 0$.

(*m*) $\sin\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$, $\cos\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$ (n is odd)

GENERAL SOLUTION OF EQUATION a cos θ + b sin θ = c

Consider, $a \sin\theta + b \cos\theta = c$ ...(*i*)

$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (*i*) has the solution only if $|c| \le \sqrt{a^2+b^2}$ let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$ and $\phi = \tan^{-1}\frac{b}{a}$ by introducing this auxiliary argument ϕ, equation (*i*) reduces to $\sin(\theta+\phi) = \frac{c}{\sqrt{a^2+b^2}}$.

GENERAL SOLUTION OF EQUATION OF FORM

$a_0 \sin^n x + a_1 \sin^{n-1} x \cos x + a_2 \sin^{n-2} \times \cos^2 x + \ldots + a_n \cos^n x = 0$ $a_0, a_1, \ldots a_n$ are real numbers.

Such an equation is solved by dividing equation by $\cos^n x$.

IMPORTANTS TIPS

(*a*) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(*b*) Avoid squaring the equations, if possible, because it may lead to extraneous solutions.

(*c*) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(*d*) The answer should not contain such values of θ, which make any of terms undefined or infinite.

(*e*) Check that denominator is not zero at any stage while solving equations.

(*f*) (*i*) If tan θ or sec θ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.

(*ii*) If cot θ or cosec θ is involved in the equation, θ should not be integral multiple of π or 0.

(g) If two different trigonometric ratios such as tan θ and sec θ are involved then after solving we cannot apply the usual formulae for general solution because periodicity of the functions are not same.

(h) If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different value of θ, then solution does not exist.

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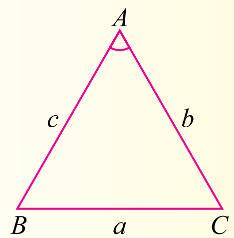
CHAPTER 5

Solutions of Triangles

1. Sine Rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.



2. Cosine Formula:

(*i*) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(*ii*) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

(*iii*) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

3. Projection Formula:

(*i*) $a = b \cos C + c \cos B$

(*ii*) $b = c \cos A + a \cos C$

(*iii*) $c = a \cos B + b \cos A$

4. Napier's Analogy - tangent rule:

(*i*) $\tan \dfrac{B - C}{2} = \dfrac{b - c}{b + c} \cot \dfrac{A}{2}$

(*ii*) $\tan \dfrac{C - A}{2} = \dfrac{c - a}{c + a} \cot \dfrac{B}{2}$

(*iii*) $\tan \dfrac{A - B}{2} = \dfrac{a - b}{a + b} \cot \dfrac{C}{2}$

5. Trigonometric Functions of Half Angles:

(i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$;

$\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

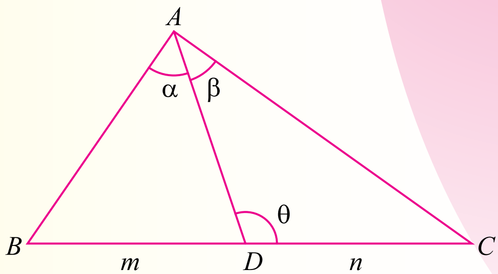
(iii) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\Delta}{s(s-a)}$ where $s=\frac{a+b+c}{2}$ is semi perimetre of triangle.

(iv) $\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2\Delta}{bc}$

6. Area of Triangle (Δ)

$$\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\sqrt{s(s-a)(s-b)(s-c)}.$$

7. m-n Rule



If $BD:DC=m:n$, then

$(m+n)\cot\theta=m\cot\alpha-n\cot\beta$

$\qquad\qquad=n\cot B-m\cot C$

8. Radius of Circumcircle

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

9. Radius of The Incircle:

(i) $r = \dfrac{\Delta}{s}$

(ii) $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$

(iii) $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ and so on

(iv) $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

10. Radius of The Ex-Circles:

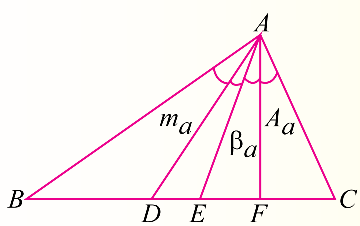
(i) $r_1 = \dfrac{\Delta}{s-a}$; $r_2 = \dfrac{\Delta}{s-b}$; $r_3 = \dfrac{\Delta}{s-c}$

(ii) $r_1 = s\tan\dfrac{A}{2}$; $r_2 = s\tan\dfrac{B}{2}$; $r_3 = s\tan\dfrac{C}{2}$

(iii) $r_1 = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ and so on.

(iv) $r_1 = 4R\sin\dfrac{A}{2}\cdot\cos\dfrac{B}{2}\cdot\cos\dfrac{C}{2}$

11. Length of Angle Bisectors, Medians and Altitudes:



(i) Length of an angle bisector from the angle $A = \beta_a = \dfrac{2\,bc\cos\frac{A}{2}}{b+c}$.

(ii) Length of median from angle $A = m_a = \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$.

(iii) Length of altitude from the angle $A = A_a = \dfrac{2\Delta}{a}$.

12. The Distances of The Special Points from Vertices and Sides of Triangle

(i) Circumcentre (O) : $OA = R$ and $O_a = R\cos A$

(ii) Incentre (I) : $IA = r\,\text{cosec}\,\dfrac{A}{2}$ and $I_a = r$

(iii) Excentre (I_1) : $I_1A = r_1\,\text{cosec}\,\dfrac{A}{2}$

(iv) Orthocentre : $HA = 2R\cos A$ & $H_a = 2R\cos B\cos C$

(v) Centroid (G) : $GA = \dfrac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$ and $G_a = \dfrac{2\Delta}{3a}$

13. Orthocentre and Pedal Triangle

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.

(i) Its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

(ii) Its sides are $a\cos A = R\sin 2A$,

$b\cos B = R\sin 2B$ and

$c\cos C = R\sin 2C$

(iii) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

Where P is orthocenter of ΔABC.

14. Excentral Triangle

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

(i) ΔABC is the pedal triangle of the $\Delta I_1I_2I_3$.

(ii) Its angles are $\dfrac{\pi}{2} - \dfrac{A}{2}$, $\dfrac{\pi}{2} - \dfrac{B}{2}$ and $\dfrac{\pi}{2} - \dfrac{C}{2}$.

(*iii*) Its sides are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$.

(*iv*) $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

(*v*) Incentre I or ΔABC is the orthocentre of the excentral $\Delta I_1I_2I_3$.

15. Distance Between Special Points

(*i*) Distance between circumcentre and orthocentre

$$OH^2 = R^2(1 - 8\cos A\cos B\cos C)$$

(*ii*) Distance between circumcentre and incentre

$$OI^2 = R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = R^2 - 2Rr$$

(*iii*) Distance between circumcentre and centroid

$$OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

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CHAPTER 6

Complex Numbers

IOTA

$$i = \sqrt{-1},\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$$

So, $$i^{4n+1} = i,\ i^{4n+2} = -1,\ i^{4n+3} = -i,\ i^{4n+4} = 1$$

In other words, $$i^n = \begin{cases} (-1)^{n/2}, \text{ if } n \text{ is an even integer} \\ (-1)^{\frac{n-1}{2}} \cdot i, \text{ if } n \text{ is an odd integer} \end{cases}.$$

The Complex Number System

$z = a + ib$, then $a - ib$ is called conjugate of z and is denoted by $\bar{z}$.

Equality in Complex Number

$z_1 = z_2 \quad \Rightarrow \quad \text{Re}(z_1) = \text{Re}(z_2)$ and $I_m(z_1) = I_m(z_2)$.

CONJUGATE COMPLEX

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

NOTES

(i) $z + \bar{z} = 2\,\text{Re}(z)$

(ii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iii) $z\bar{z} = a^2 + b^2$ which is real

(iv) If z is purely real then $z - \bar{z} = 0$

(v) If z is purely imaginary then $z + \bar{z} = 0$

IMPORTANT PROPERTIES OF CONJUGATE

(*a*) $\overline{(\bar{z})} = z$

(*b*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(*c*) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(*d*) $\overline{z_1 \, z_2} = \bar{z}_1 \cdot \bar{z}_2$

(*e*) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}\,; z_2 \neq 0$

(*f*) If $f(\alpha + i\beta) = x + iy \quad \Rightarrow \quad f(\alpha - i\beta) = x - iy$

IMPORTANT PROPERTIES OF MODULUS

(*a*) $|\,z\,| \geq 0$

(*b*) $|\,z\,| \geq \text{Re}(z)$

(*c*) $|\,z\,| \geq \text{Im}\,(z)$

(*d*) $|\,z\,| = |\,\bar{z}.\,| = |-z| = |-\bar{z}.|$

(*e*) $z\bar{z}. = |z|^2$

(*f*) $|z_1\,z_2| = |z_1| \cdot |z_2|$

(*g*) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|},\ z_2 \neq 0$

(*h*) $|\,z^n\,| = |\,z\,|^n$

(*i*) $|\,z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}\,(z_1\,\bar{z}_2)$

or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|\cos\,(\theta_1 - \theta_2)$

(*j*) $|\,z_1 + z_2|^2 + |\,z_1 - z_2|^2 = 2\,[|\,z_1|^2 + |\,z_2|^2]$

(*k*) $||\,z_1| - |\,z_2|| \leq |\,z_1 + z_2| \leq |\,z_1| + |\,z_2|$ **[Triangle Inequality]**

(*l*) $||\,z_1| - |\,z_2|| \leq |\,z_1 - z_2| \leq |\,z_1| + |\,z_2|$ **[Triangle Inequality]**

(*m*) If $\left|z + \dfrac{1}{z}\right| = a\ (a > 0)$, then $\max|\,z\,| = \dfrac{a + \sqrt{a^2 + 4}}{2}$

and $\min|\,z\,| = \dfrac{1}{2}\left(\sqrt{a^2 + 4} - a\right)$.

IMPORTANT PROPERTIES OF AMPLITUDE

(*a*) $\text{amp}\,(z_1 \cdot z_2) = \text{amp}\,z_1 + \text{amp}\,z_2 + 2\,k\pi\,;\ k \in I.$

(*b*) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}\,z_1 - \text{amp}\,z_2 + 2\,k\pi\,;\ k \in I.$

(*c*) $\text{amp}\,(z^n) = n\,\text{amp}(z) + 2k\pi$, where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(*d*) $\log\,(z) = \log\,(re^{i\theta}) = \log r + i\theta = \log|z| + i\,\text{amp}\,(z)$.

Demoivre's Theorem

Case I: If n is any integer then

(*i*) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(*ii*) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots$
$(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \theta_3 + \ldots\theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots\theta_n)$

Case II: If $p, q \in Z$ and $q \neq 0$ then $(\cos\theta + i\sin\theta)^{p/q}$

$$= \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where $k = 0, 1, 2, 3 \ldots q - 1$.

Cube Root of Unity

(*i*) The cube roots of unity are $1, \dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$.

(*ii*) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(*c*) $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$a^3 + b^3 = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$

$x^2 + x + 1 = (x - \omega)(x - \omega^2)$

SQUARE ROOT OF COMPLEX NUMBER

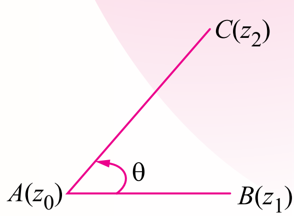
$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right\} \text{ for } b > 0$$

and $$\pm\left\{\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right\} \text{ for } b < 0 \text{ where } |z| = \sqrt{a^2+b^2}.$$

ROTATION

$$\frac{z_2 - z_0}{|z_2 - z_0|} = \frac{z_1 - z_0}{|z_1 - z_0|}e^{i\theta}$$

Take θ in anticlockwise direction.



RESULT RELATED WITH TRIANGLE

(a) Equilateral triangle:

$\Rightarrow$ $$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

or $$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

(b) Area of triangle ΔABC given by modulus of $\frac{1}{4}\begin{vmatrix} z_1 & \overline{z}_1 & 1 \\ z_2 & \overline{z}_2 & 1 \\ z_3 & \overline{z}_3 & 1 \end{vmatrix}$.

EQUATION OF LINE THROUGH POINTS z_1 AND z_2

$$\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z}_1 & 1 \\ z_2 & \overline{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\overline{z}_1 - \overline{z}_2) + z_1\overline{z}(z_2 - z_1) + \overline{z}_2 - \overline{z}_1 z_2 = 0$$

$$\Rightarrow \quad z(\overline{z}_1 - \overline{z}_2)i + \overline{z}(z_2 - z_1)i + i(z_1\overline{z}_2 - \overline{z}_1 z_2) = 0$$

Let $(z_2 - z_1)i = a$, then equation of line is $\boxed{\overline{a}z + a\overline{z} + b = 0}$ where $a \in C$ & $b \in R$.

NOTES

(i) Complex slope of line $\overline{a}z + a\overline{z} + b = 0$ is $-a\frac{1}{\overline{a}}$.

(ii) Two lines with slope μ_1 and μ_2 are parallel or perpendicular if $\mu_1 = \mu_2$ or $\mu_1 + \mu_2 = 0$.

(iii) Length of perpendicular from point $A(\alpha)$ to line $\overline{a}z + a\overline{z} + b = 0$ is $\frac{|\overline{a}\alpha + a\overline{\alpha} + b|}{2|a|}$.

EQUATION OF CIRCLE

(a) Circle whose centre is z_0 and radii $= r$

$$|z - z_0| = r$$

(b) General equation of circle

$$z\overline{z} + a\overline{z} + \overline{a}z + b = 0$$

centre '$-a$' & radii $= \sqrt{|a|^2 - b}$

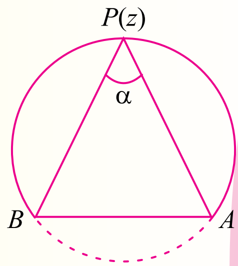
(*c*) Diameter form $(z-z_1)(\overline{z}-\overline{z}_2)+(z-z_2)(\overline{z}-\overline{z}_1)=0$

or $$\arg\left(\frac{z-z_1}{z-z_2}\right)=\pm\frac{\pi}{2}$$

(*d*) Equation $\left|\frac{z-z_1}{z-z_2}\right| = k$ represent a circle if $k \neq 1$ and a straight line if $k=1$.

(*e*) Equation $|z-z_1|^2+|z-z_2|^2=k$

represent circle if $k \geq \frac{1}{2}|z_1-z_2|^2$



(*f*) $\arg\left(\frac{z-z_1}{z-z_2}\right)=\alpha \quad 0<\alpha<\pi, \alpha \neq \frac{\pi}{2}$

represent a segment of circle passing through $A(z_1)$ and $B(z_2)$.

STANDARD LOCI

(*a*) $|z-z_1|+|z-z_2|=2k$ (a constant) represent

(*i*) If $2k>|z_1-z_2|$ $\Rightarrow$ An ellipse

(*ii*) If $2k=|z_1-z_2|$ $\Rightarrow$ A line segment

(*iii*) If $2k<|z_1-z_2|$ $\Rightarrow$ No solution

(*b*) Equation $||z-z_1|-|z-z_2||=2k$ (a constant) represent

(*i*) If $2k<|z_1-z_2|$ $\Rightarrow$ A hyperbola

(*ii*) If $2k=|z_1-z_2|$ $\Rightarrow$ A line ray

(*iii*) If $2k>|z_1-z_2|$ $\Rightarrow$ No solution

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CHAPTER 7

Quadratic Equation

SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS

(a) The solutions of the quadratic equation, $ax^2 + bx + c = 0$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(b) The expression $b^2 - 4\,ac = D$ is called the discriminant of the quadratic equation.

(c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then;

(i) $\alpha + \beta = -b/a$

(ii) $\alpha\beta = c/a$

(iii) $|\alpha - \beta| = \sqrt{D}/|a|$

(d) Quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ *i.e.*

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ *i.e.* x^2 –(sum of roots)x + product of roots = 0.

Nature of Roots

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then;

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal).

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + i\,q$ is one root of a quadratic equation, then the other root must be the conjugate $p - i\,q$ & vice versa.

($p, q \in R$ & $i = \sqrt{-1}$).

(*b*) Consider the quadratic equation $ax^2 + bx + c = 0$

where $a, b, c \in Q$ & $a \neq 0$ then;

(*i*) If D is a perfect square, then roots are rational.

(*ii*) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$. (if a, b, c are rational)

Common Roots of Two Quadratic Equations

(*a*) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then $a\,\alpha^2 + b\,\alpha + c = 0$ & $a'\,\alpha^2 + b'\,\alpha + c' = 0$. By Cramer's Rule

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

Therefore, $\alpha = \dfrac{ca' - c'a}{ab' - a'b} = \dfrac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is

$(ca' - c'a)^2 = (ab' - a'b)\,(bc' - b'c)$

(*b*) If both roots are same then $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

Roots Under Particular Cases

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(*a*) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign

(*b*) If $c = 0 \Rightarrow$ one roots is zero other is $-b/a$

(*c*) If $a = c \Rightarrow$ roots are reciprocal to each other

(*d*) If $\left.\begin{matrix} a > 0\ c < 0 \\ a < 0\ c > 0 \end{matrix}\right\} \Rightarrow$ roots are of opposite signs.

(*e*) If $\left.\begin{matrix} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are negative.

(*f*) If $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are positive.

(*g*) If sign of a = sign of $b \neq$ sign of $c \Rightarrow$ Greater root in magnitude is negative.

(*h*) If sign of b = sign of $c \neq$ sign of $a \Rightarrow$ Greater root in magnitude is positive.

(*i*) If $a+b+c=0 \Rightarrow$ one root is 1 and second root is c/a.

MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSION

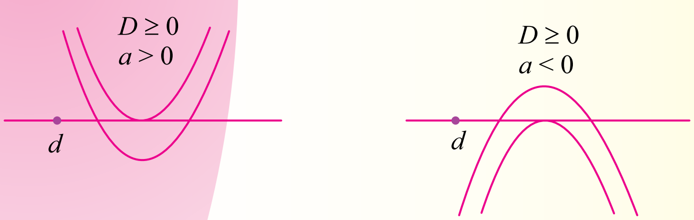
Maximum & Minimum Values of expression $y = ax^2 + bx + c$ is $\frac{-D}{4a}$ which occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

$$y \in \left[\frac{-D}{4a}, \infty\right) \text{ if a} > 0 \quad \& \quad y \in \left(-\infty, \frac{-D}{4a}\right] \text{ if a} < 0.$$

LOCATION OF ROOTS

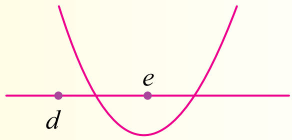
Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$

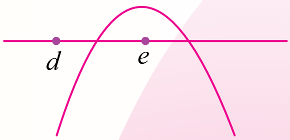
(*a*) Conditions for both the roots of f(x) = 0 to be greater than a specified number 'd' are $\boldsymbol{D \geq 0; a.f(d) > 0}$ **&** $\boldsymbol{(-b/2a) > d}$.



(*b*) Conditions for the both roots of $f(x) = 0$ to lie on either side of the number 'd' in other words the number 'd' lies between the roots of $f(x) = 0$ is $\boldsymbol{a.f(d) < 0}$.

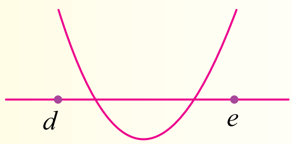
(c) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d,e) *i.e.*, $d < x < e$ is $\boldsymbol{f(d).f(e) < 0}$.

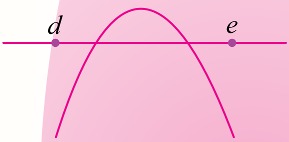




(d) Conditions that both roots of $f(x) = 0$ to be confined between the numbers d & e are (here $d < e$).

$D \geq 0$; $a.f(d) > 0$ & $af(e) > 0$; $d < (-b/2a) < e$





GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES

$f(x, y) = ax^2 + 2\,hxy + by^2 + 2gx + 2\,fy + c$ may be resolved into two linear factors if;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ OR } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

THEORY OF EQUATIONS

If $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ are the roots of the equation;

$f(x) = a_0\,x^n + a_1\,x^{n-1} + a_2\,x^{n-2} + ... + a_{n-1}\,x + a_n = 0$ where $a_0, a_1, ...a_n$ are constants $a_0 \neq 0$ then,

$$\sum\alpha_1 = -\frac{a_1}{a_0}, \sum\alpha_1\alpha_2 = +\frac{a_2}{a_0}, \sum\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0},$$

$$\alpha_1\alpha_2\alpha_3.........\alpha_n = (-1)^n\frac{a_n}{a_0}$$

NOTES

(i) Every odd degree equation has at least one real root whose sign is opposite to that of its last term, when coefficient of highest degree term is (+)ve {If not then make it (+)ve}.

Ex. $x^3 - x^2 + x - 1 = 0$

(ii) Even degree polynomial whose last term is (–)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+)ve & one (–)ve.

(iii) If equation contains only even power of x & all coefficient are (+)ve, then all roots are imaginary.

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CHAPTER 8

Permutation and Combination

FUNDAMENTAL PRINCIPLE OF COUNTING (Counting without actually counting)

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of different ways of

(a) Simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events (known as multiplication principle).

(b) Happening of exactly one of the events is $m + n$ (known as addition principle).

FACTORIAL

A Useful Notation : $n! = n(n-1)(n-2) \ldots\ldots 3 \cdot 2 \cdot 1;$

$$n! = n \,.\, (n-1)! \text{ where } n \in W$$

$$0! = 1! = 1$$

$$(2n)! = 2^n \,.\, n!\, [1.\ 3.\ 5.\ 7 \ldots (2n-1)]$$

PERMUTATION

(a) ${}^nP_r$ denotes the number of permutations of n different things, taken r at a time ($n \in N, r \in W, n \geq r$)

$${}^nP_r = n(n-1)(n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

(b) The number of permutations of n things taken all at a time when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining

$n-(p+q+r)$ are all different is : $\dfrac{n!}{p!\,q!\,r!}$.

(c) The number of permutation of n different objects taken r at a time, when a particular object is always to be included is $r! \cdot {}^{n-1}C_{r-1}$.

(d) The number of permutation of n different object taken r at a time, when repetition be allowed any number of times is $n \times n \times n \ldots r$ times $= n^r$.

(e) **(i)** The number of circular permutations of n different things taken all at a time is ; $(n-1)! = \dfrac{{}^nP_n}{n}$.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is $\dfrac{(n-1)!}{2}$.

(ii) The number of circular permutation of n different things taking r at a time distinguishing clockwise & anticlockwise arrangement is $\dfrac{{}^nP_r}{r}$.

COMBINATION

(a) ${}^nC_r$ denotes the number of combinations of n different things taken r at a time, and ${}^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{{}^nP_r}{r!}$ where $r \le n$; $n \in N$ and $r \in W$. ${}^nC_r$ is also denoted by $\dbinom{n}{r}$ or A_r^n or $C(n, r)$.

(b) The number of combination of n different things taking r at a time.

(i) when p particular things are always to be included $= {}^{n-p}C_{r-p}$.

(ii) when p particular things are always to be excluded $= {}^{n-p}C_r$.

(iii) when p particular things are always to be included and q particular things are to be excluded $= {}^{n-p-q}C_{r-p}$.

(c) Given n different objects, the number of ways of selecting atleast one of them is, ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(d) **(i)** Total number of ways in which it is possible to make a selection by taking some or all out of $p + q + r + \ldots$ things, where p are alike of one kind, q a like of a second kind, r alike of third kind and so on is given by : $(p+1)(q+1)(r+1) \ldots - 1$.

(ii) The total number of ways of selecting one or more things from p identical of one kind, q identical things of second kind, r identical things of third kind and n different things is $(p + 1)(q + 1)(r + 1)$ $2^n - 1$.

DIVISORS

Let $N = p^a \cdot q^b \cdot r^c$ ... where p, q, r ... are distinct primes and a, b, c ... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is

$= (a + 1)(b + 1)(c + 1)$ ....

(b) The sum of these divisors is $= (p^0 + p^1 + p^2 + ... + p^a)$

$(q^0 + q^1 + q^2 + ... + q^b)(r^0 + r^1 + r^2 + ... + r^c)$ ...

(c) Number of ways in which N can be resolved as a product of two factor is =

$\frac{1}{2}(a + 1)(b + 1)(c + 1)$ ... if N is not a perfect square

$\frac{1}{2}[(a + 1)(b + 1)(c + 1) ... + 1]$ if N is a perfect square

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

DIVISION AND DISTRIBUTION

(a) **(i)** The number of ways in which $(m + n)$ different things can be divided into two groups containing m & n things respectively is :

$$\frac{(m+n)!}{m!\,n!}\ (m \neq n).$$

(ii) If $m = n$, it means the groups are equal & in this case the number of subdivision is $\frac{(2n)!}{n!\,n!\,2!}$; for in any one way it is possible to inter change the two groups without obtaining a new distribution.

(iii) If $2n$ things are to be divided equally between two persons then the number of ways $= \frac{(2n)!}{n!\,n!\,(2!)} \times 2!$.

(b) **(i)** Number of ways in which $(m + n + p)$ different things can be divided into three groups containing m, n & p things respectively is $\dfrac{(m+n+p)!}{m!\,n!\,p!}$, $m \neq n \neq p$.

(ii) If $m = n = p$ then the number of groups $= \dfrac{(3n)!}{n!\,n!\,n!\,3!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\dfrac{(3n)!}{(n!)^3}$.

(c) In general, the number of ways of dividing n distinct object into l groups containing p objects each, m groups containing q objects each is equal to $\dfrac{n!\,(l+m)!}{(p!)^l\,(q!)^m\,l!\,m!}$.

Here $lp + mq = n$.

(d) Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them $= p^n$.

(e) Number of ways in which n identical may be distributed among p persons if each person may receive one, one or more things is; ${}^{n+p-1}C_n$.

DEARRANGEMENT

Number of ways in which n letters can be placed in n directed envelopes so that no letter goes into its own envelope is

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}\right]$$

IMPORTANT RESULT

(a) Number of rectangle of any size in a square of size $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of square of any size is $\sum_{r=1}^{n} r^2$.

(b) Number of rectangle of any size in a rectangle of size $n \times p$ $(n < p)$ is $\dfrac{np}{4}(n+1)(p+1)$ and number of squares of any size is $\sum_{r=1}^{n}(n+1-r)$ $(p+1-r)$.

(c) If there are n points in a plane of which m ($< n$) are collinear :

(i) Total number of lines obtained by joining these points is ${}^nC_2 - {}^mC_2 + 1$.

(ii) Total number of different triangle ${}^nC_3 - {}^mC_3$.

(d) Maximum number of point of intersection of n circles is ${}^nP_2$ and n lines is ${}^nC_2$.

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CHAPTER 9

Binomial Theorem

IMPORTANT TERMS IN THE BINOMIAL EXPANSION ARE

(*a*) **General term:** The general term or the (r + 1)th term in the expansion of $(x + y)^n$ is given by

$$T_{r+1} = {}^nC_r\, X^{n-r} \cdot y^r$$

(*b*) **Middle term:** The middle term (s) is the expansion of $(x + y)^n$ is (are):

(*i*) If n is even, there is only one middle term which is given by

$$T_{(n+2)/2} = {}^nC_{n/2} \cdot x^{n/2} \cdot y^{n/2}$$

(*ii*) If n is odd, there are two middle terms which are

$$T_{(n+1)/2} \text{ and } T_{[(n+1)/2]+1}$$

(*c*) **Term independent of x:** Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

IF $(\sqrt{A} + B)^n = I + f$, WHERE I & n ARE POSITIVE INTEGERS AND $0 \le f < 1$, THEN

(*d*) $(I + f) \cdot f = K^n$ if n is odd & $A - B^2 = K > 0$

(*b*) $(I + f)(1 - f) = k^n$ if n is even & $\sqrt{A} - B < 1$

SOME RESULTS ON BINOMIAL COEFFICIENTS

(*a*) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(*b*) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

(c) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} \ldots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + \ldots = C_n = 2^n$

(f) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

(g) $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!n!}$

(h) $C_0 \cdot C_r + C_1 . C_{r+1} + C_2 \cdot C_{r+2} + \ldots + C_{n-r} C_n = \frac{(2n)!}{(n+r)!(n-r)}$

GREATEST COEFFICIENT AND GREATEST TERM IN EXPANSION OF $(x + a)^n$

(a) If n is even greatest coefficient is ${}^nC_{n/2}$.

If n is odd greatest coefficient is ${}^nC_{\left(\frac{n-1}{2}\right)}$ or ${}^nC_{\left(\frac{n+1}{2}\right)}$

(b) **For greatest term :** Greatest term

$$= \begin{cases} T_p \text{ and } T_{p+1} & \text{if } \dfrac{n+1}{\left|\dfrac{x}{a}\right|+1} \text{ is an integer} \\ T_{q+1} & \text{if } \dfrac{n+1}{\left|\dfrac{x}{a}\right|+1} \text{ is non integer and} \in (q, q+1),\ q \in I \end{cases}$$

Multinomial Theorem

For any $n \in N$,

(i) $(x_1 + x_2 + \ldots + x_k)^n = \sum_{r_1 + r_2 + \ldots + r_k = n} \frac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$

(ii) The general term in the above expansion is

$$\frac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$$

Here total number of terms in the expansion $= {}^{n+k-1}C_{k-1}$.

BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES

If $n \in Q$, then $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots \infty$ provided $|x| < 1$.

NOTES

(i) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots \infty$

(ii) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots \infty$

(iii) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \infty$

(iv) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \infty$

EXPONENTIAL SERIES

(a) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots \infty$; where x may be any real or complex number and $e = \lim\limits_{n \to \infty}\left(1 + \dfrac{1}{n}\right)^n$.

(b) $a^x = 1 + \dfrac{x}{1!}\ell na + \dfrac{x^2}{2!}\ell n^2 a + \dfrac{x^3}{3!}\ell n^3 a + \ldots \infty$, where $a > 0$.

LOGARITHMIC SERIES

(a) $ln\,(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots \infty$, where $-1 < x \le 1$.

(b) $ln\,(1 - x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots \infty$, where $-1 \le x < 1$.

(c) $ln\,\dfrac{(1 + x)}{(1 - x)} = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots \infty\right)$, $|x| < 1$.

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CHAPTER 10

Sequence and Series

ARITHMETIC PROGRESSION (A.P.)

(*a*) n^{th} term of this A.P. $T_n = a + (n-1)d$, where $d = T_n - T_{n-1}$

(*b*) The sum of the first n terms: $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l]$ where l is the last term.

(*c*) Also n^{th} term $T_n = S_n - S_{n-1}$

NOTES

(*i*) Sum of first n terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in n, in such case the common difference is twice the coefficient of n^2. i.e. $2A$.

(*ii*) n^{th} term of an A.P. is of the form $An + B$ i.e. a linear expression in n, in such case the coefficient of n is the common difference of the A.P. i.e. A.

(*iii*) Three numbers in A.P. can be taken as $a - d, a, a + d$; four numbers in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$ five numbers in A.P. are $a - 2d, a - d, a, a + d, a + 2d$ and six terms in A.P. are $a - 5d$, $a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(*iv*) If for A.P. p^{th} term is q, q^{th} term is p, then r^{th} term is $= p + q - r$ and $(p + q)^{th}$ term is 0.

(*v*) If a_1, a_2, a_3 ... and b_1, b_2, b_3 ... are two A.P.s, then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3$ ... are also in A.P.

(*vi*) (*a*) If each term of an A.P. is increased or decreased by the same number, then the resulting sequence is also an A.P. having the same common difference.

(*b*) If each term of an A.P. is multiplied or divided by the same non zero number (k), then the resulting sequence is also an A.P. whose common difference is kd and d/k respectively, where d is common difference of original A.P.

(*vii*) Any term of an A.P. (except the first and last) is equal to half the sum of terms which are equidistant from it.

$$T_r = \frac{T_{r-k} + T_{r+k}}{2}, \quad k < r$$

GEOMETRIC PROGRESSION (G.P.)

(*a*) n^{th} term $T_n = ar^{n-1}$

(*b*) Sum of the first n terms $S_n = \dfrac{a(r^n - 1)}{r - 1}$, if $r \neq 1$

(*c*) Sum of infinite G.P. when $|r| < 1$ and $n \to \infty$, $r^n \to 0$

$$S_\infty = \frac{a}{1-r}; \; |r| < 1$$

(*d*) Any 3 consecutive terms of a G.P. can be taken as a/r, a, ar; any 4 consecutive terms of a G.P. can be taken as a/r^3, a/r, ar, ar^3 and so on.

(*e*) If a, b, c are in G.P. $\Rightarrow b^2 = ac \Rightarrow \log a, \log b, \log c$, are in A.P.

NOTES

(*i*) In an G.P. product of k^{th} term from beginning and k^{th} term from the last is always constant which equal to product of first term and last term.

(*ii*) Three number in G.P. : $a/r, a, ar$

Five numbers in G.P. : $a/r^2, a/r, a, ar, ar^2$

Four numbers in G.P. : $a/r^3, a/r, ar, ar^3$

Six numbers in G.P. : $a/r^5, a/r^3, a/r, ar, ar^3, ar^5$

(*iii*) If each term of a G.P. be raised to the same power, then resulting series is also a G.P.

(*iv*) If each term of a G.P. be multiplied or divided by the same non-zero quantity, then the resulting sequence is also a G.P.

(*v*) If a_1, a_2, a_3 ... and b_1, b_2, b_3, ... be two G.P.'s of common ratio r_1 and r_2 respectively, then a_1b_1, a_2b_2 ... and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ ... will also form a G.P. common ratio will be $r_1\,r_2$ and $\frac{r_1}{r_2}$ respectively.

(*vi*) In a positive G.P. every term (except first) is equal to square root of product of its two terms which are equidistant from it.

i.e., $T_r = \sqrt{T_{r-k}\,T_{r+k}},\ k < r.$

(*vii*) If a_1, a_2, a_3 ... a_n is a G.P. of non-zero, non-negative terms, then $\log a_1, \log a_2$, ... $\log a_n$ is an A.P. and vice-versa.

HARMONIC PROGRESSION (H.P.)

Note: No term of any H.P. can be zero. If a, b, c are in H.P.

$$\Rightarrow b = \frac{2ac}{a+c} \quad \text{or} \quad \frac{a}{c} = \frac{a-b}{b-c}$$

MEANS

(*a*) **Arithmetic Mean (AM):** If a, b are any two given numbers and $a, A_1, A_2, \ldots A_n, b$ are in A.P. then $A_1, A_2, \ldots A_n$ are the n AM's between a and b, then $A_1 = a + d, A_2 = a + 2d, \ldots, A_n = a + nd$, where $d = \frac{b-a}{n+1}$.

Note: Sum of n AM's inserted between a and b is equal to n times the single AM between a and b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a and b.

(*b*) **Geometric Mean (GM):** If a, b, c are in GP, b is the GM between a and c, $b^2 = ac$, therefore $b = \sqrt{ac}$.

***n*-geometric means between two numbers:** If a, b are two given positive numbers and $a, G_1, G_2, \ldots, G_n, b$ are in GP then $G_1, G_2, G_3, \ldots G_n$ are n GM's between a and b.

$G_1 = ar, G_2 = ar^2, \ldots G_n = ar^n$, where $r = (b/a)^{1/n+1}$

Note: The product of n GM's between a and b is equal to n^{th} power of the single GM between a and b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a and b.

(c) Harmonic Mean (HM): If a, b, c are in HP, then b is HM between a and c, then $b = \dfrac{2ac}{a+c}$.

NOTES

(i) If A, G, H, are respectively AM, GM, HM between two positive number a and b then

(a) $G^2 = AH (A, G, H$ constitute a $GP)$

(b) $A \geq G \geq H$

(c) $A = G = H \Rightarrow a = b$

(ii) Let $a_1, a_2, \ldots, a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as

$$A = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

$$G = (a_1\, a_2 \ldots a_n)^{1/n} \quad \text{and} \quad H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots \frac{1}{a_n}\right)}$$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = \ldots = a_n$.

ARITHMETICO-GEOMETRIC SERIES

Sum of First n terms of an Arithmetico-Geometric Series:

Let $S_n = a + (a + d)r + (a + 2d)r^2 + \ldots + [a + (n - 1)d]r^{n-1}$

then $S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{[a+(n-1)d]}{1-r}, \ r \neq 1$

Sum of infinity

If $|r| < 1$ and $n \to \infty$ then $\lim\limits_{n\to\infty} r^n = 0 \Rightarrow S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

SIGMA NOTATIONS

Theorems:

(a) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(b) $\sum_{r=1}^{n} ka_r = k\sum_{r=1}^{n} a_r$

(c) $\sum_{r=1}^{n} k = nk$; where k is a constant.

RESULTS

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

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CHAPTER 11

Straight Line

❖ **Distance Formula:** $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

❖ **Section Formula:** $x = \dfrac{mx_2 \pm nx_1}{m \pm n}$; $y = \dfrac{my_2 \pm ny_1}{m \pm n}$

❖ **Centroid, Incentre & Excentre:**

Centroid $G\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$,

Incentre $I\left(\dfrac{ax_1 + bx_2 + cx_3}{a + b + c}, \dfrac{ay_1 + by_2 + cy_3}{a + b + c}\right)$

Excentre $I_1\left(\dfrac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \dfrac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$

Remarks:

(i) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter, coincides.

(ii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining. Orthocentre and circumcentre in the ratio 2 : 1.

(ii) In a isosceles triangle centroid, orthocentre, incentre, circumcentre lies on the same line.

AREA OF TRIANGLE

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left| \frac{1}{2} \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \right|$$

EQUATION OF STRAIGHT LINE

(*a*) Equation of *a* line parallel to x-axis at *a* distance a is $y = a$ or $y = -a$.

(*b*) Equation of x-axis is $y = 0$.

(*c*) Equation of line parallel to y-axis at a distance b is $x = b$ or $x = -b$.

(*d*) Equation of y-axis is $x = 0$.

If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line $AB = \dfrac{y_2 - y_1}{x_2 - x_1}$.

STANDARD FORMS OF EQUATIONS OF A STRAIGHT LINE

(*a*) **Slope Intercept form :** Let m be the slope of a line and c its intercept on y-axis, then the equation of this straight line is written as : $y = mx + c$.

(*b*) **Point Slope form :** If m be the slope of a line and it passes through a point (x_1, y_1), then its equation is written as : $y - y_1 = m(x - x_1)$.

(*c*) **Two point form :** Equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(*d*) **Intercept form :** If a and b are the intercepts made by a line on the axes of x and y, its equation is written as : $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(*e*) **Normal form :** If p is the length of perpendicular on a line from the origin and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as :

$x \cos \alpha + y \sin \alpha = p$ (p is always positive), where $0 \leq \alpha < 2\pi$.

(*f*) **Parametric form :** $\dfrac{x-h}{\cos\theta} = \dfrac{y-k}{\sin\theta} = r$ is the equation.

(*g*) **General form :** We know that a first degree equation in x and y, $ax + by + c = 0$ always represents a straight line. This form is known as general form of straight line.

(*i*) Slope of this line $= \dfrac{-a}{b} = \dfrac{\text{coefficient of } x}{\text{coefficient of } y}$

(*ii*) Intercept by this line on x-axis $= -\dfrac{c}{a}$ and intercept by this line on y-axis $= -\dfrac{c}{b}$.

(*iii*) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2+b^2}$.

ANGLE BETWEEN TWO LINES

(*a*) If θ be the angle between two lines : $y = m_1x + c_1$ and $y = m_2x + c_2$, then

$\tan\theta = \pm\left(\dfrac{m_1 - m_2}{1 + m_1m_2}\right)$.

(*b*) If equation of lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then these line are–

(*i*) Parallel $\Leftrightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

(*ii*) Perpendicular $\Leftrightarrow a_1a_2 + b_1b_2 = 0$

(*iii*) Coincident $\Leftrightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

(*iv*) Intersecting $\Leftrightarrow \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

LENGTH OF PERPENDICULAR FROM A POINT ON A LINE

Length of perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$ is

$= \left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$.

In particular the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \dfrac{|c|}{\sqrt{a^2 + b^2}}$.

DISTANCE BETWEEN TWO PARALLEL LINES

(a) The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.

(**Note :** The coefficients of x & y in both equations should be same).

(b) The area of the parallelogram $= \dfrac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distance between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1 x + c_1$, $y = m_1 x + c_2$ and $y = m_2 x + d_1$, $y = m_2 x + d_2$ is given $\left|\dfrac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

EQUATION OF LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE

(a) Equation of line parallel to line $ax + by + c = 0$.

$$ax + by + \lambda = 0$$

(b) Equation of line perpendicular to line $ax + by + c = 0$.

$$bx - ay + k = 0$$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

STRAIGHT LINE MAKING A GIVEN ANGLE WITH A LINE

Equations of lines passing through a poing (x_1, y_1) and making an angle α, with the line $y = mx + c$ is written as :

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

POSITION OF TWO POINTS WITH RESPECT TO A GIVEN LINE

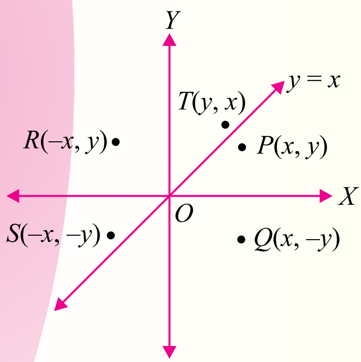
Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

CONCURRENCY OF LINES

Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

REFLECTION OF A POINT

Let $P(x, y)$ be any point, then its image with respect to



(a) x-axis is $Q(x, -y)$

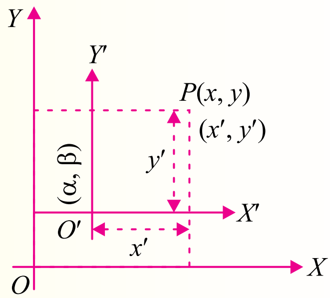
(b) y-axis is $R(-x, y)$

(c) origin is $S(-x, -y)$

(d) line $y = x$ is $T(y, x)$

TRANSFORMATION OF AXES

(a) **Shifting of origin without rotation of axes :** If coordinates of any point $P(x, y)$ with respect to new origin (a, b) will be (x', y')

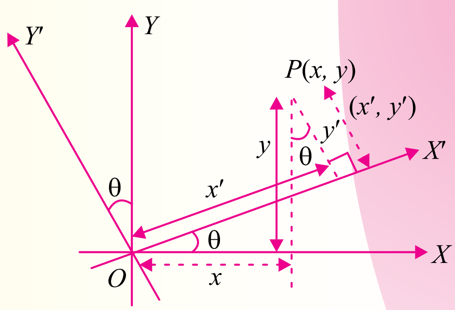


then $x = x' + \alpha, \quad y = y' + \beta$

or $\quad x' = x - \alpha, \quad y' = y - \beta$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

(*b*) Rotation of axes without shifting the origin : Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX' and OY', where $\angle X'OX = \angle YOY' = \theta$



then $x = x' \cos\theta - y' \sin\theta$

$\quad y = x' \cos\theta - y' \cos\theta$

or $\quad y' = x \cos\theta + y \sin\theta$

$\quad y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

New \ Old	$x\downarrow$	$y\downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES

If equation of two intersecting lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then equation of bisectors of the angles between these lines are written are:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \qquad ...(1)$$

(*a*) Equation of bisector of angle containing origin : If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of bisectors of the angle containing the origin is obtained by taking sign in (1).

(*b*) Equation of bisector of acute/obtuse angles : See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by –1 to make the constant terms positive.

Determinate the sign of $a_1a_2 + b_1b_2$

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sing in eq. (1)	use – sign in eq. (1)
–	use – sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

FAMILY OF LINES

If equation of two lines be $P = a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional information given in the problem.

GENERAL EQUATION AND HOMOGENEOUS EQUATION OF SECOND DEGREE

(a) A general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of straight lines if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$

(b) If θ be the angle between the lines, then $\tan\theta = \pm\dfrac{2\sqrt{h^2 - ab}}{a+b}$.

Obviously these lines are

(i) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$.

(ii) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(c) Homogeneous equation of 2nd degree $ax^2 + 2hxy + by^2 = 0$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right) x \equiv y = m_1 x \text{ and } y = m_2 x$$

and $\quad m_1 + m_2 = -\dfrac{2h}{b}$; $m_1 m_2 = \dfrac{a}{b}$

These straight lines passes through the origin and for finding the angle between these lines same formula as given for general equation is used.

The condition that these lines are :

(i) At right angles to each other is $a + b = 0$. i.e. co-efficient of x^2 + co-efficient of $y^2 = 0$.

(ii) Coincident is $h^2 = ab$.

(iii) Equally inclined to the axis of x is $h = 0$. i.e. coefficient of $xy = 0$.

(d) The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by

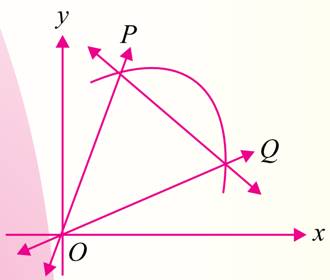
$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}, \; a \neq b, \; h \neq 0.$$

(e) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

(*f*) If lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel then distance between them is $= 2\sqrt{\dfrac{g^2 - ac}{a(a+b)}}$.

EQUATIONS OF LINES JOINING THE POINTS OFINTERSECTION OF A LINE AND A CURVE TO THE ORIGIN

Let the equation of curve be:



$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad ...(i)$$

and straight line be

$$lx + my + n = 0 \qquad ...(ii)$$

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$$

STANDARD RESULTS

(*a*) Area of rhombus formed by lines $a|x| + b|y| + c = 0$ or $\pm ax \pm by + c = 0$ is $\dfrac{2c^2}{|ab|}$.

(*b*) Area of triangle formed by line $ax + by + c = 0$ and axes is $\dfrac{c^2}{2|ab|}$.

(*c*) Co-ordinate of foot of perpendicular (h, k) from (x_1, y_1) to the line $ax + by + c = 0$ is given by $\dfrac{h - x_1}{a} = \dfrac{k - y_1}{b} = \dfrac{-(ax_1 + by_1 + c)}{a^2 + b^2}$.

(*d*) Image of point (x_1, y_1) w.r. to the line $ax + by + c = 0$ is given by

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}.$$

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CHAPTER 12

Circles

STANDARD EQUATIONS OF THE CIRCLE

(*a*) **Central Form:** If (h, k) is the centre and r is the radius of the circle then its equation is $\boldsymbol{(x-h)^2+(y-k)^2=r^2}$.

(*b*) **General equation of circle:** $\boldsymbol{x^2+y^2+2gx+2fy+c=0}$, where g, f, c are constants and centre is $(-g, -f)$

i.e. $\left(-\dfrac{\text{coefficient of } x}{2}, -\dfrac{\text{coefficient of } y}{2}\right)$

and radius $r=\sqrt{g^2+f^2-c}$

Intercepts cut by the circle on axes

The intercepts cut by the circle $x^2+y^2+2gx+2fy+c=0$ on:

(*i*) x-axis $=2\sqrt{g^2-c}$

(*ii*) y-axis $=2\sqrt{f^2-c}$

Diameter form of circle

$\boldsymbol{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0}$

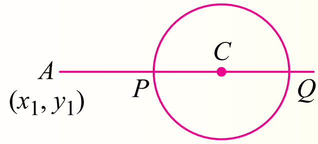
The parametric forms of the circle

(*i*) The parametric equation of the circle $x^2+y^2=r^2$ are $x=r\cos\theta$, $y=r\sin\theta$; $\theta\in[0, 2\pi]$.

(*ii*) The parametric equation of the circle $(x-h)^2+(y-k)^2=r^2$ is $x=h+r\cos\theta$, $y=k+r\sin\theta$ where θ is parameter.

POSITION OF A POINT W.R.T. CIRCLE

(*a*) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1, y_1) then:



Point (x_1, y_1) lies outside the circle or on the circle or inside the circle according as

$\Rightarrow \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$ or $S_1 >, =, < 0$

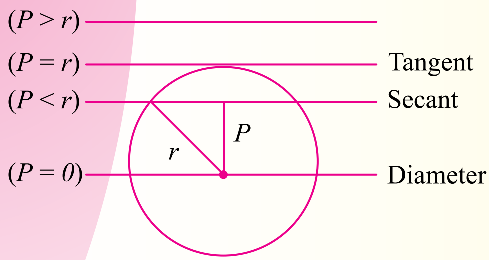
(*b*) The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $AC - r$ respectively.

(*c*) The power of point is given by S_1.

TANGENT LINE OF CIRCLE

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(*a*) Condition of Tangency: The line $L = 0$ touches the circle $S = 0$ if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. $P = r$.



(*b*) Equation of the tangent (T = 0)

(*i*) Tangent at the point (x_1, y_1) on the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(*ii*) (1) The tangent at the point $(a \cos t, a \sin t)$ on the circle $x^2 + y^2 = a^2$ is $x \cos t + y \sin t = a$.

(2) The point of intersection of the tangents at the points

$$P(\alpha) \text{ and } Q(\beta) \text{ is } \left(\frac{a \cos \frac{\alpha+\beta}{2}}{\cos \frac{\alpha-\beta}{2}}, \frac{a \sin \frac{\alpha+\beta}{2}}{\cos \frac{\alpha-\beta}{2}} \right).$$

(*iii*) The equation of tangent at the point (x_1, y_1) on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

(*iv*) If line $y = mx + c$ is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and contact points are $\left(\pm \frac{am}{\sqrt{1+m^2}}, \pm \frac{1}{\sqrt{1+m^2}}\right)$ or $\left(\pm \frac{a^2 m}{c}, \pm \frac{a^2}{c}\right)$ and equation of tangent is $y = mx \pm a\sqrt{1+m^2}$

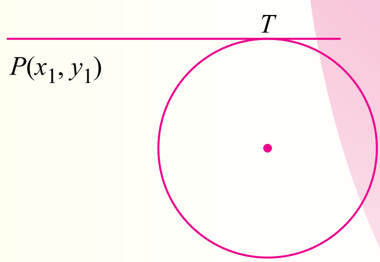
(*v*) The equation of tangent with slope m of the circle $(x - h)^2 + (y - k)^2 = a^2$ is $(y - k) = m(x - h) \pm a\sqrt{1+m^2}$

Note: To get the equation of tangent at the point (x_1, y_1) on any curve we replace xx_1 in place x^2, yy_1 in place of y^2, $\frac{x + x_1}{2}$ in place of x, $\frac{y + y_1}{2}$ in place of y, $\frac{xy_1 + yx_1}{2}$ in place of xy and c in place of c.

(*c*) **Length of tangent ($\sqrt{S_1}$):** The length of tangent drawn from point (x_1, y_1) out side the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is,}$$

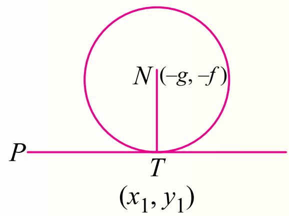
$$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$



(*d*) **Equation of Pair of tangents ($SS_1 = T^2$):** $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$ or $SS_1 = T^2$.

NORMAL OF CIRCLE

(*a*) Equation of normal at point (x_1, y_1) of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is



$$y - y_1 = \left(\frac{y_1 + f}{x_1 + g}\right)(x - x_1)$$

(b) The equation of normal on any point (x_1, y_1) of circle $x^2 + y^2 = a^2$ is $\left(\frac{y}{x} = \frac{y_1}{x_1}\right)$.

CHORD OF CONTACT

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. $T = 0$ same as equation of tangent).

EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$)

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

DIRECTOR CIRCLE

Equation of director circle is $x^2 + y^2 = 2a^2$.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

Note: The director circle of

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is } x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0.$$

POLE AND POLAR

The equation of the polar is the $T = 0$, so the polar of point (x_1, y_1) w.r.t. circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Pole of a given line with respect to a circle

Similar terms we can get the coordinates of the pole. The pole of

$$lx + my + n = 0$$

w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-la^2}{n}, \frac{-ma^2}{n}\right)$.

FAMILY OF CIRCLES

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K\,S_2 = 0$ $(K \neq -1)$.

(b) The equation of the family of cirlces passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form:

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$ where K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m\,(x - x_1)] = 0$, where K is a parameter.

(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda\,L_2L_3 + \mu\,L_3L_1 = 0$ provided coefficient of $xy = 0$ & coefficient of x^2 = coefficient of y^2.

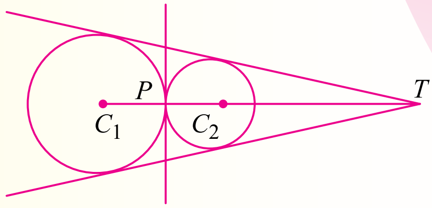
(f) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $L_1L_3 + \lambda\,L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

DIRECT AND TRANSVERSE COMMON TANGENTS

Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

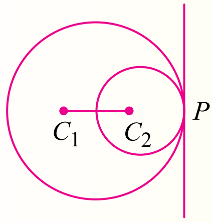
(a) Both circles will touch

(i) **Externally** if $C_1C_2 = r_1 + r_2$, point P divides C_1C_2 in the ratio $r_1 : r_2$ (internally).



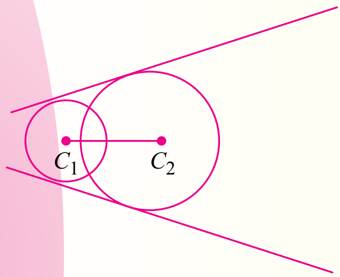
In this case there are **three common tangents**.

(*ii*) Internally if $C_1C_2 = |r_1 - r_2|$, point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be only **one common tangent**.

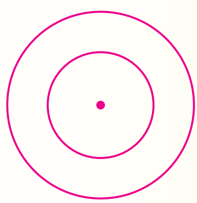


(*b*) The circles will intersect

when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents**.

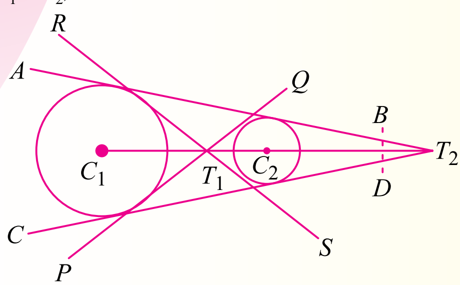


(*c*) The circles will not intersect



(*i*) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$. In this case there will be no common tangent.

(*ii*) $C_1C_2 > (r_1 + r_2)$



Note: Length of direct common tangent = $\sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent = $\sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

THE ANGLE OF INTERSECTION OF TWO CIRCLES

$$\cos\theta = \frac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}} \text{ or } \boxed{\cos\theta = \left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)}$$

the circles to be orthogonal is

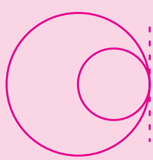
$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

RADICAL AXIS OF THE TWO CIRCLES ($S_1 - S_2 = 0$)

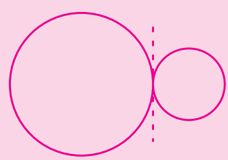
Then the equation of radical axis is given by $S_1 - S_2 = 0$.

NOTES

(*i*) If two circles touches each other then common tangent is radical axis.

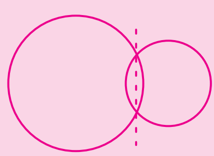


$S_1 - S_2 = 0$



$S_1 - S_2 = 0$

(*ii*) If two circles cuts each other then common chord is radical axis.



$S_1 - S_2 = 0$

(*iii*) If two circles cuts third circle orthogonally then radical axis of first two is locus of centre of third circle.

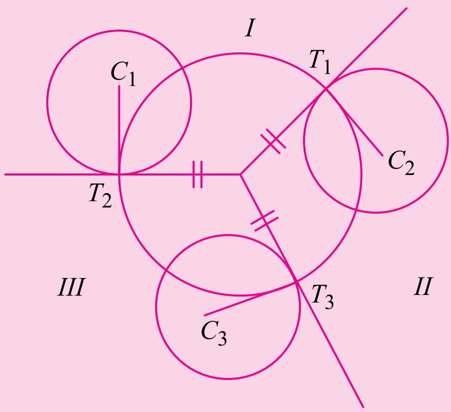
(*iv*) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

Radical centre

The radical centre of three circles is the point from which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles.

NOTES

(i) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.



(ii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.

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CHAPTER 13

Parabola

GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is:

$$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2$$

$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Case (i) When the focus lines on the directrix

In this case $D \equiv abc + 2\,fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if:

$e > 1$, $h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1$, $h^2 = ab$ the lines will coincident.

$e < 1$, $h^2 < ab$ the lines will be imaginary.

When the focus does not lie on the directix

The conic represents:

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1$; $D \neq 0$ $h^2 = ab$	$0 < e < 1$; $D \neq 0$ $h^2 < ab$	$D \neq 0$; $e > 1$ $h^2 > ab$	$e > 1$; $D \neq 0$ $h^2 > ab$; $a + b = 0$

Standard equation of a parabola is $y^2 = 4ax$. For this parabola:

(i) Vertex is $(0, 0)$

(ii) Focus is $(a, 0)$

(iii) Axis is $y = 0$

(iv) Directrix is $x + a = 0$

Latus rectum

A focal chord perpendicular to the axis of a parabola is called the LATUS RECTUM. For $y^2 = 4ax$.

(*i*) Length of the latus rectum = $4a$.

(*ii*) Length of the semi latus rectum = $2a$.

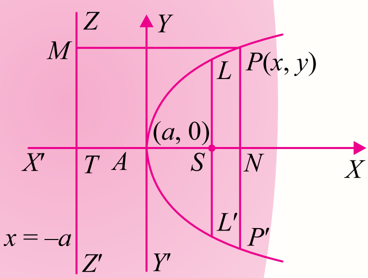
(*iii*) Ends of the latus rectum are $L(a, 2a)$ & $L'(a, -2a)$.

PARAMETRIC REPRESENTATION

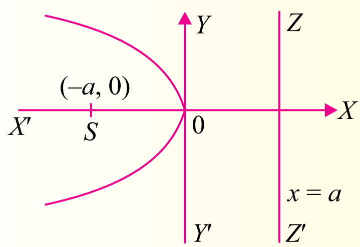
The simplest & the best form of representing the co-ordinates of a point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$. The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

TYPES OF PARABOLA

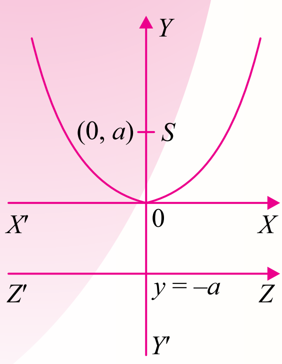
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$.



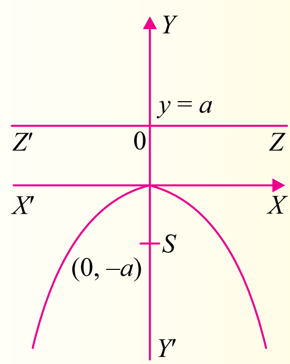
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Para-metric equation	Focal length
$y^2 = 4ax$	(0, 0)	$(a, 0)$	$y = 0$	$x = -a$	$4a$	$(a, \pm 2a)$	$(at^2, 2at)$	$x + a$
$y^2 = -4ax$	(0, 0)	$(-a, 0)$	$y = 0$	$x = a$	$4a$	$(-a, \pm 2a)$	$(-at^2, 2at)$	$x - a$
$x^2 = +4ay$	(0, 0)	$(0, a)$	$x = 0$	$y = -a$	$4a$	$(\pm 2a, a)$	$(2at, at^2)$	$y + a$
$x^2 = -4ay$	(0, 0)	$(0, -a)$	$x = 0$	$y = a$	$4a$	$(\pm 2a, -a)$	$(2at, -at^2)$	$y - a$
$(y - k)^2 = 4a(x - h)$	(h, k)	$(h + a, k)$	$y = k$	$x + a - h = 0$	$4a$	$(h + a, k \pm 2a)$	$(h + at^2, k + 2at)$	$x - h + a$
$(x - p)^2 = 4b(y - q)$	(p, q)	$(p, b + q)$	$x = p$	$y + b - q = 0$	$4b$	$(p \pm 2a, q + a)$	$(p + 2at, q + at^2)$	$y - q + b$

POSITION OF A POINT RELATIVE TO A PARABOLA

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1{}^2 - 4ax_1$ is positive, zero or negative.

CHORD JOINING TWO POINTS

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.

NOTES

(*i*) If PQ is focal chord then $t_1t_2 = -1$.

(*ii*) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

(*iii*) If $t_1t_2 = k$ then chord always passes a fixed point $(-ka, 0)$.

LINE & A PARABOLA

(*a*) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a > = < cm$

$\Rightarrow$ condition of tangency is, $c = \dfrac{a}{m}$.

Note: Line $y = mx + c$ will be tangent to parabola

$x^2 = 4ay$ if $\boldsymbol{c = -am^2}$.

(*b*) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is : $\left(\dfrac{4}{m^2}\right)\sqrt{a(1 + m^2)(a - mc)}$.

Note: length of the focal chord making an angle α with the x-axis is $4a \operatorname{cosec}^2 \alpha$.

TANGENT TO THE PARABOLA $y^2 = 4ax$:

(*a*) **Point form:** Equation of tangent to the given parabola at its point (x_1, y_1) is $yy_1 = 2a(x + x_1)$.

(*b*) **Slope form:** Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(*c*) **Parametric form:** Equation of tangent to the given parabola at its point $P(t)$, is–

$$ty = x + at^2$$

Note: Point of intersection of the tangents at the point t_1 & t_2 is $[at_1 t_2, a(t_1 + t_2)]$. (i.e. G.M. and A.M. of abscissae and ordinates of the points).

NORMAL TO THE PARABOLA $y^2 = 4ax$

(*a*) **Point form:** Equation of normal to the given parabola at its point (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

(*b*) **Slope form:** Equation of normal to the given parabola whose slope is 'm', is $y = mx - 2am - am^3$ foot of the normal is $(am^2, -2am)$.

(*c*) **Parametric form:** Equation of normal to the given parabola at its point $P(t)$, is $y + tx = 2at + at^3$.

NOTES

(*i*) Point of intersection of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1 t_2 + 2), -at_1 t_2 (t_1 + t_2))$.

(*ii*) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(*iii*) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1 t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

CHORD OF CONTACT

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.

Remember that the area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$. Also note that the chord of contact exists only if the point P is not inside.

CHORD WITH A GIVEN MIDDLE POINT

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$.

DIAMETER

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is $y = 2a/m$, where m = slope of parallel chords.

CONORMAL POINTS

Foot of the normals of three concurrent normals are called conormals point.

(*i*) Algebraic sum of the slopes of three concurrent normals of parabola $y^2 = 4ax$ is zero.

(*ii*) Sum of ordinates of the three conormal points on the parabola $y^2 = 4ax$ is zero.

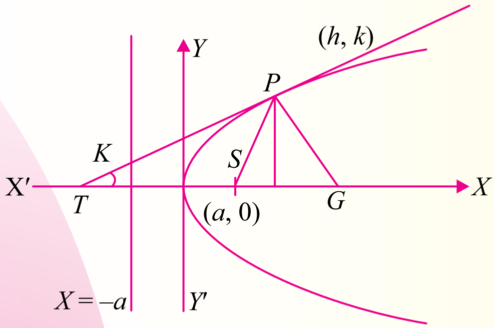
(*iii*) Centroid of the triangle formed by three co-normal points lies on the axis of parabola.

(*iv*) If $27ak^2 < 4(h - 2a)^3$ satisfied then three real and distinct normal are drawn from point (h, k) on parabola $y^2 = 4ax$.

(*v*) If three normals are drawn from point $(h, 0)$ on parabola $y^2 = 4ax$, then $h > 2a$ and one the normal is axis of the parabola and other two are equally inclined to the axis of the parabola.

IMPORTANT HIGHLIGHTS

(*a*) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then $ST = SG = SP$ where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.



(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix**, and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point $P(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of a length $a\sqrt{1+t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean beetween segments of any focal chord

i.e. $2a = \dfrac{2bc}{b+c}$ or $\dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{a}$.

(f) Image of the focus lies on directrix with respect to any tangent of parabola $y^2 = 4ax$.

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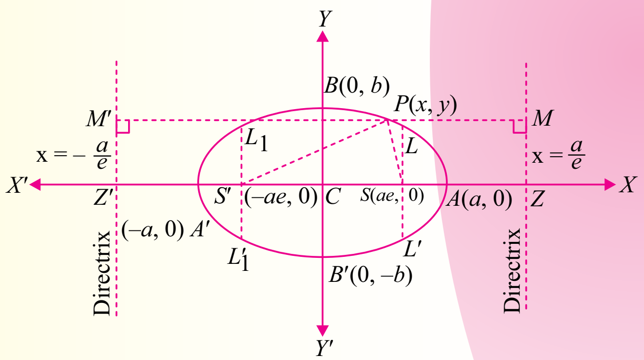
CHAPTER 14

Ellipse

The co-ordinate axis is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Where $a > b$ & $b^2 = a^2(1 - e^2)$

$\Rightarrow \quad a^2 - b^2 = a^2 e^2.$

where e = eccentricity $(0 < e < 1)$.



FOCI : S = (ae, 0) & S′ ≡ (– ae, 0).

(*j*) **Latus Rectum:** The focal chord perpendicular to the major axis is called the **latus rectum**.

(*i*) Length of latus rectum

$$(LL') = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1 - e^2)$$

(*ii*) Equation of latus rectum : $x = \pm\, ae$.

(*iii*) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, -\frac{b^2}{a}\right)$,

$L_1\left(-ae, \frac{b^2}{a}\right)$ and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

(*k*) Focal radii: SP = $a - ex$ and $S'P$ = a + ex

$\Rightarrow \quad SP + S'P = 2a$ = Major axis.

(*l*) Eccentricity: $e = \sqrt{1 - \frac{b^2}{a^2}}$

POSITION OF A POINT W.R.T. AN ELLIPSE

The point $P(x_1, y_1)$ lines outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > < \text{or} = 0$.

PARAMATRIC REPRESENTATION

The equations x = a cos θ & y = b sin θ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where θ is a parameter (eccentric angle).

Note that if P(θ) ≡ (a cos θ, b sin θ) is on the ellipse then ;
Q(θ) ≡ (a cos θ, a sin θ) is on the auxiliary circle.

LINE AND AN ELLIPSE

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two real points, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$.

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$.

TANGENT TO THE ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(*a*) Point form: Equation of tangent to the given ellipse at its point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

(*b*) **Slope form:** Equation of tangent to the given ellipse whose slope is 'm', $y = mx \pm \sqrt{a^2m^2 + b^2}$.

Point of contact are $\left(\dfrac{\pm a^2 m}{\sqrt{a^2m^2 + b^2}}, \dfrac{\pm b^2}{\sqrt{a^2m^2 + b^2}}\right)$

(*c*) **Parametric form:** Equation of tangent to the given ellipse at its point $(a \cos \theta, b \sin \theta)$, is $\dfrac{x \cos \theta}{a} + \dfrac{y \sin \theta}{b} = 1$.

NORMAL TO THE ELLIPSE $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(*a*) **Point form:** Equation of the normal to the given ellipse at (x_1, y_1) is $\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$.

(*b*) **Slope form:** Equation of a normal to the given ellipse whose slope is 'm' is $y = mx \pm \dfrac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

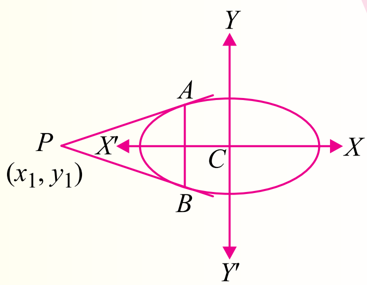
(*c*) **Parametric form:** Equation of the normal to the given ellipse at the point $(a \cos \theta, b \sin \theta)$ is $ax. \sec \theta - by. \operatorname{cosec} \theta = (a^2 - b^2)$.

CHORD OF CONTACT

If PA and PB be the tangents from point $P(x_1, y_1)$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then the equation of the chord of contact AB is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ or $T = 0$ at (x_1, y_1).

PAIR OR TANGENTS

If $P(x_1, y_1)$ be any point lies outside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and a pair of tangents PA, PB can be drawn to it from P.



Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1, \quad T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

i.e., $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$

DIRECTOR CIRCLE

$x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

EUQATION OF CHORD WITH MID POINT (x_1, y_1)

i.e. $\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right) = \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)$

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CHAPTER 15

Hyperbola

- Standard equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$

  where $b^2 = a^2(e^2 - 1)$

  or $a^2e^2 = a^2 + b^2$ *i.e.* $e^2 = 1 + \frac{b^2}{a^2}=1+\left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

(*a*) Foci:

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(*b*) Equations of Directrices:

$x=\frac{a}{e}$ & $x=-\frac{a}{e}$.

(*c*) Vertices:

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

(*d*) Latus Rectum:

(*i*) Equation: $x = \pm ae$

(*ii*) Length $=\frac{2b^2}{a}=\frac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})}=2a(e^2-1)$

$= 2e$(distance from focus to directrix)

(*iii*) Ends: $\left(ae,\frac{b^2}{a}\right),\left(ae,\frac{-b^2}{a}\right);\left(-ae,\frac{b^2}{a}\right),\left(-ae,\frac{-b^2}{a}\right)$

(*f*) Focal Property:

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis *i.e.* $||PS| - |PS'|| = 2a$. The distance SS' = focal length.

(g) Focal Distance:

Distance of any point $P(x, y)$ on hyperbola from foci $PS = ex - a$ & $PS' = ex + a$.

Conjugate Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each.

Auxillary Circle: $x^2 + y^2 = a^2$.

Parametric Representation: $x = a \sec \theta$ & $y = b \tan \theta$

Position of *A* point '*P*' w.r.t. *A* Hyperbola:

$S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > = \text{ or } < 0$ according as the point (x_1, y_1) lines inside, on or outside the curve.

Tangents:

(i) Slope Form: $y = m \times \pm \sqrt{a^2m^2 - b^2}$

(ii) Point Form: at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(iii) Parametric Form: $\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$.

❖ **Normal to The Hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(a) Point form: Equation of the normal to the given hyperbola at the point $P(x_1, y_1)$ on it is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2 = a^2e^2$.

(b) Slope form: The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{(a^2 - m^2b^2)}}$ foot of normal are $\left(\pm \frac{a^2}{\sqrt{(a^2 - m^2b^2)}}, \mp \frac{mb^2}{\sqrt{(a^2 - m^2b^2)}}\right)$.

(c) Parametric form: The equation of the normal at the point P $(a \sec \theta, b \tan \theta)$ to the given hyperbola is $\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2 = a^2e^2$.

DIRECTOR CIRCLE

Equation to the director circle is: $x^2 + y^2 = a^2 - b^2$.

CHORD OF CONTACT

If PA and PB be the tangents from point $P(x_1, y_1)$ to the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the equation of the chord of contact AB is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ or $T = 0$ at (x_1, y_1).

EQUATION OF CHORD WITH MID POINT (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whose mid-point be (x_1, y_1) is $T = S_1$ where $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$, $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

i.e. $\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right) = \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right)$.

ASYMPTOTES

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

Reflection property of the hyperbola: An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.

Rectangular or Equilateral Hyperbola: $xy = c^2$, eccentricity is $\sqrt{2}$.

Vertices: $(\pm c \pm c)$; Focii : $\left(\pm\sqrt{2}c, \pm\sqrt{2}c\right)$. Directrices : $x + y = \pm\sqrt{2}\,c$.

Latus Rectum (l): $= l = 2\sqrt{2}\,c = T.A. = C.A.$

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at $P(t)$ is $\frac{x}{t} + ty = 2c$.

Equation of the normal at $P(t)$ is $xt^3 - yt = c(t^4 - 1)$.

Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

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CHAPTER 16

Statistics

ARITHMETIC MEAN

(*i*) Arithmetic Mean for Unclassified (Ungrouped or Raw) Data: If there are n observations, $x_1, x_2, x_3, \ldots, x_n$, then their arithmetic mean

$$A \text{ or } \bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

(*ii*) Arithmetic Mean for Discrete Frequency Distribution or Ungrouped Frequency Distribution: Let $f_1, f_2, \ldots, f_n$ be corresponding frequencies of $x_1, x_2, \ldots, x_n$. Then, arithmetic mean

$$A = \frac{x_1 f_1 + x_2 f_2 + \ldots + x_n f_n}{f_1 + f_2 + \ldots + f_n} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i}$$

(*iii*) Arithmetic Mean for Classified (Grouped) Data or Grouped Frequency Distribution: For a classified data, we take the class marks $x_1, x_2, \ldots, x_n$, of the classes, then arithmetic mean by

$$A = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i}$$

Combined Mean: If $A_1, A_2, \ldots, A_r$ are means of $n_1, n_2, \ldots, n_r$ observations respectively, then arithmetic mean of the combined group is called the combined mean of the observation.

$$A = \frac{n_1A_1 + n_2A_2 + \ldots + n_rA_r}{n_1 + n_2 + \ldots + n_r} = \frac{\sum_{i=1}^{r} n_iA_i}{\sum_{i=1}^{r} n_i}$$

MEDIAN

Median for Simple Distribution or Raw Data

Firstly, arrange the data in ascending or descending order and then find the number of observations *n*.

(*a*) If n is odd, then $\left(\frac{n+1}{2}\right)$ th term is the median.

(*b*) If n is even, then there are two middle terms namely $\left(\frac{n}{2}\right)$ th and $\left(\frac{n}{2}+1\right)$ th terms, median is mean of these terms.

Median for Classified (Grouped) Data or Grouped Frequency Distribution

For a continuous distribution, median

$$M_d = l + \frac{\frac{N}{2} - C}{f} \times h$$

where, I = lower limit of the median class

f = frequency of the median class

N = total frequency = $\sum_{i=1}^{n} f_i$

C = cumulative frequency of the class just before the median class

h = length of the median class

Mode for Classified (Grouped) Distribution or Grouped Frequency Distribution

$$M_O = l + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

where, l = lower limit of the modal class

f_0 = frequency of the modal class

f = frequency of the pre-modal class

f = frequency of the post-modal class

h = length of the class interval

Relation between Mean, Median and Mode

(*i*) Mean – Mode = 3 (Mean – Median)

(*ii*) Mode = 3 Median – 2 Mean

MEAN DEVIATION (MD)

(*i*) For simple (raw) distribution, $\delta = \dfrac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$

where, n = number of terms, $\bar{x} = A$ or M_d or M_O

(*ii*) For unclassified frequency distribution, $\delta = \dfrac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i}$

(*iii*) For classified distribution, $\delta = \dfrac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i}$

where, x_i is the class mark of the interval.

STANDARD DEVIATION AND VARIANCE

(*i*) For simple distribution

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}} = \frac{1}{n}\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

where, n is a number of observations and $\bar{x}$ is mean.

(*ii*) For discrete frequency distribution

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} f(x_i - \bar{x})^2}{N}} = \frac{1}{N}\sqrt{N\sum_{i=1}^{n} f_i\, x_i^2 - \left(\sum_{i=1}^{n} f_i x_i\right)^2}$$

(*iii*) For continuous frequency distribution

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{N}}$$

where, x_i is class mark of the interval.

Standard Deviation of the Combined Series

If n_1, n_2 are the sizes, $\bar{X}_1, \bar{X}_2$ are the means and σ_1, σ_2, are the standard deviation of the series, then the standard deviation of the combined series is

$$\sigma = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1 + n_2}}$$

where, $d_1 = \bar{X}_1 - \bar{X}, d_2 = \bar{X}_2 - \bar{X}$

and $\bar{X} = \dfrac{n_1\bar{X} + n_2\bar{X}_2}{n_1 + n_2}$

IMPORTANT POINTS TO BE REMEMBERED

(*i*) The ratio of SD (σ) and the AM ($\bar{x}$) is called the coefficient of standard deviation $\left(\dfrac{\sigma}{\bar{x}}\right)$

(*ii*) The percentage form of coefficient of SD i.e. $\left(\dfrac{\sigma}{\bar{x}}\right) \times 100$ is called coefficient of variation.

(*iii*) The distribution for which the coefficient of variation is less is more consistent.

(*iv*) Standard deviation of first n natural numbers is $\sqrt{\dfrac{n^2 - 1}{12}}$.

(*v*) Standard deviation is independent of change of origin, but it depends on change of scale.

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CHAPTER 17

Inverse Trigonometric Functions

PRINCIPAL VALUES AND DOMAINS OF INVERSE TRIGONOMETRIC/CIRCULAR FUNCTIONS

Function			Domain	Range
(*i*)	$y = \sin^{-1} x$	where	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(*ii*)	$y = \cos^{-1} x$	where	$-1 \le x \le 1$	$0 \le y \le \pi$
(*iii*)	$y = \tan^{-1} x$	where	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(*iv*)	$y = \text{cosec}^{-1} x$	where	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2};\ y \ne 0$
(*v*)	$y = \sec^{-1} x$	where	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi;\ y \ne \frac{\pi}{2}$
(*vi*)	$y = \cot^{-1} x$	where	$x \in R$	$0 < y < \pi$

PROPERTIES OF INVERSE CIRCULAR FUNCTIONS

P-1:

(*i*) $y = \sin(\sin^{-1} x) = x,\ x \in [-1, 1],\ y \in [-1, 1]$, y is aperiodic.

(*ii*) $y = \cos(\cos^{-1} x) = x,\ x \in [-1, 1],\ y \in [-1, 1]$, y is aperiodic.

(*iii*) $y = \tan(\tan^{-1} x) = x,\ x \in R,\ y \in R$, y is aperiodic.

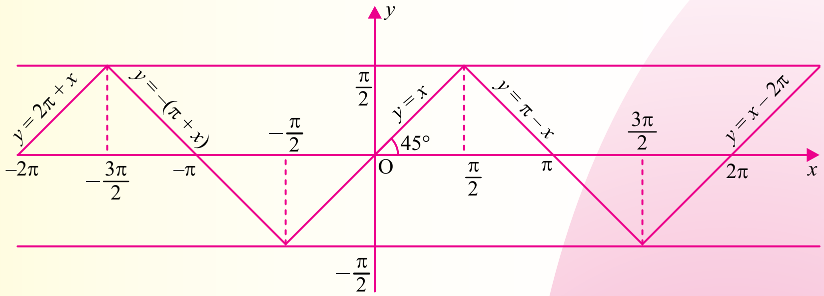
(*iv*) $y = \cot(\cot^{-1} x) = x,\ x \in R,\ y \in R$, y is aperiodic.

(*v*) $y = \text{cosec}(\text{cosec}^{-1} x) = x,\ |x| \ge 1,\ |y| \ge 1$, y is aperiodic.

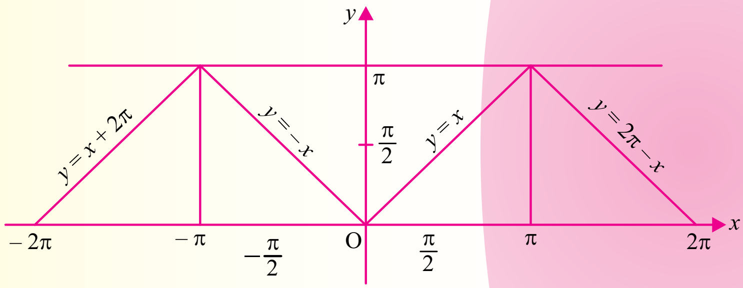
(*vi*) $y = \sec(\sec^{-1} x) = x,\ |x| \ge 1;\ |y| \ge 1$, y is aperiodic.

P-2:

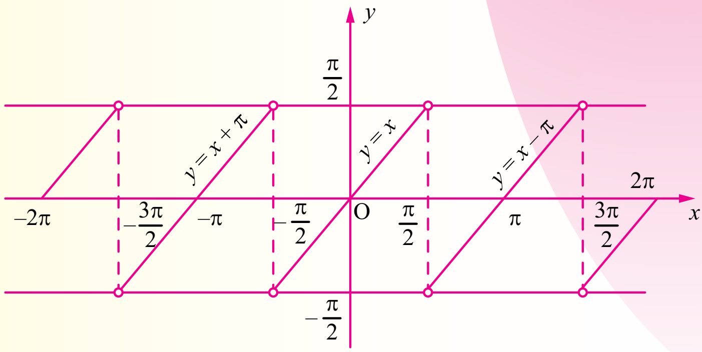
(i) $y = \sin^{-1}(\sin x), x \in R, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Periodic with period 2π.



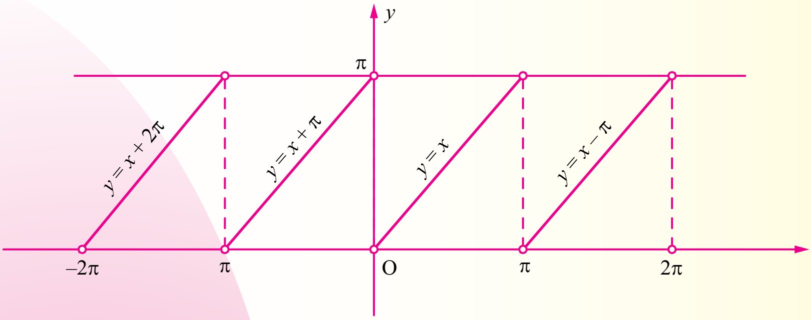
(ii) $y = \cos^{-1}(\cos x), x \in R, y \in [0, \pi]$, periodic with period 2π.



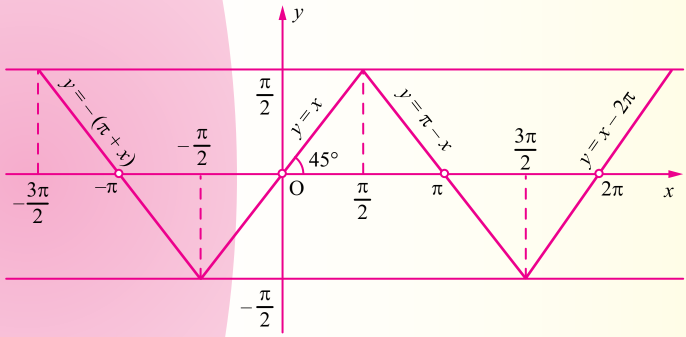
(iii) $y = \tan^{-1}(\tan x)$



(*iv*) $y = \cot^{-1}(\cot x), x \in R - \{n\pi\}, y \in (0, \pi)$, periodic with period π.

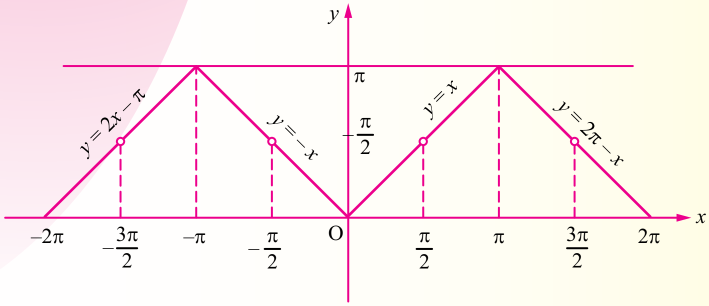


(*v*) $y = \text{cosec}^{-1}(\text{cosec } x), x \in R - \{n\pi, n \in I\}$ $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ y, is periodic with period 2π.



(*vi*) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π

$$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}, y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$



P-3:

(*i*) $\operatorname{cosec}^{-1} x = \sin^{-1} \dfrac{1}{x}$; $x \le -1, x \ge 1$

(*ii*) $\sec^{-1} x = \cos^{-1} \dfrac{1}{x}$; $x \le -1, x \ge 1$

(*iii*) $\cot^{-1} x = \tan^{-1} \dfrac{1}{x}$; $x > 0$

$= \pi + \tan^{-1} \dfrac{1}{x}$; $x < 0$

P-4:

(*i*) $\sin^{-1}(-x) = -\sin^{-1} x, -1 \le x \le 1$

(*ii*) $\tan^{-1}(-x) = -\tan^{-1} x, x \in R$

(*iii*) $\cos^{-1}(-x) = \pi - \cos^{-1} x, -1 \le x \le 1$

(*iv*) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$

(*v*) $\sec^{-1}(-x) = \pi - \sec^{-1} x, x \le -1$ or $x \ge 1$

(*vi*) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, x \le -1$ or $x \ge 1$

P-5:

(*i*) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$ $\quad -1 \le x \le 1$

(*ii*) $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$ $\quad x \in R$

(*ii*) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}$ $\quad |x| \ge 1$

P-6:

(*i*) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy}$ where $x > 0, y > 0$ & $xy < 1$

$= \pi + \tan^{-1} \dfrac{x+y}{1-xy}$, where $x > 0, y > 0$ & $xy > 1$

$= \dfrac{\pi}{2}$, where $x > 0, y > 0$ & $xy = 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \dfrac{x-y}{1+xy}$ where $x > 0, y > 0$

(iii) $\sin^{-1} x + \sin^{-1} y = \sin^{-1} [x\sqrt{1-y^2} + y\sqrt{1-x^2}]$,

where $x > 0, y > 0$ & $(x^2 + y^2) < 1$

Note that: $x^2 + y^2 < 1 \Rightarrow 0 < \sin^{-1} x + \sin^{-1} y < \dfrac{\pi}{2}$

(iv) $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} [x\sqrt{1-y^2} + y\sqrt{1-x^2}]$,

where $x > 0, y > 0$ and $x^2 + y^2 > 1$.

Note that: $x^2 + y^2 > 1 \Rightarrow \dfrac{\pi}{2} < \sin^{-1} x + \sin^{-1} y < \pi$.

(v) $\sin^{-1} x - \sin^{-1} y = \sin^{-1} [x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ where $x > 0, y > 0$.

(vi) $\cos^{-1} x + \cos^{-1} y = \cos^{-1} [xy - \sqrt{1-x^2}\sqrt{1-y^2}]$, where $x > 0, y > 0$

(vii) $\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); x < y,\ x, y > 0 \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); x > y,\ x, y > 0 \end{cases}$

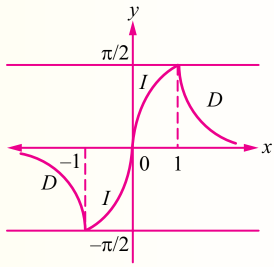
(viii) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1} \left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$

if $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$.

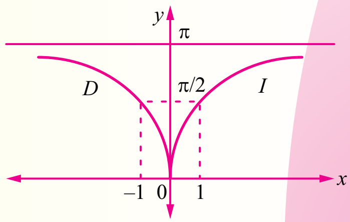
Note that: In the above results x & y are taken positive. In case if these are given as negative, we first apply P-4 and then use above results.

SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS

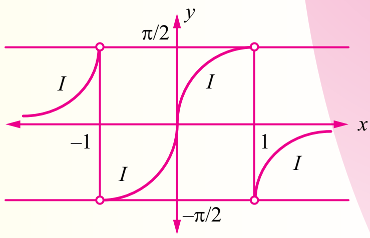
(a) $y = f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \left[\begin{array}{lll} 2\tan^{-1} x & \text{if} & |x| \le 1 \\ \pi - 2\tan^{-1} x & \text{if} & x > 1 \\ -(\pi + 2\tan^{-1} x) & \text{if} & x < -1 \end{array}\right.$



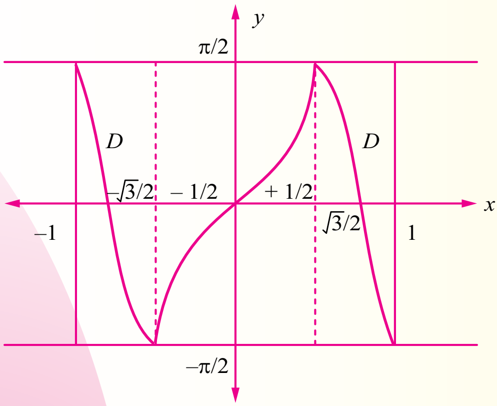
(*b*) $y = f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x \text{ if } x \geq 0 \\ -2\tan^{-1} x \text{ if } x < 0 \end{cases}$



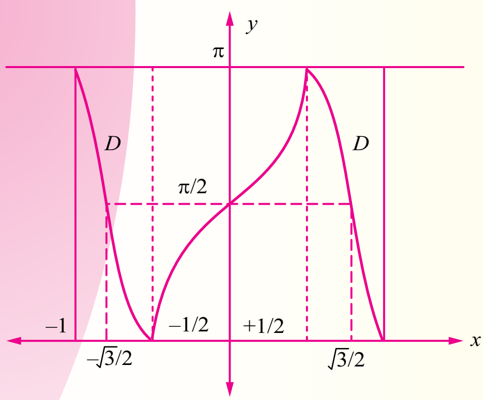
(*c*) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1} x & \text{if} \quad |x| < 1 \\ \pi + 2\tan^{-1} x & \text{if} \quad x < -1 \\ -(\pi - 2\tan^{-1} x) & \text{if} \quad x > 1 \end{cases}$



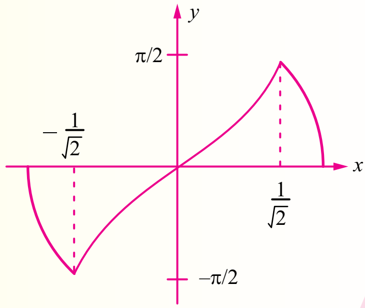
(*d*) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1} x) & \text{if} \quad -1 \leq x \leq -\dfrac{1}{2} \\ 3\sin^{-1} x & \text{if} \quad -\dfrac{1}{2} \leq x \leq \dfrac{1}{2} \\ \pi - 3\sin^{-1} x & \text{if} \quad \dfrac{1}{2} \leq x \leq 1 \end{cases}$



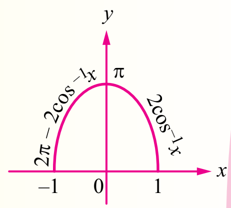
(*e*) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x \le -\dfrac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1}x & \text{if } \dfrac{1}{2} \le x \le 1 \end{cases}$



(*f*) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} -(\pi + 2\sin^{-1}x) & -1 \le x \le -\dfrac{1}{\sqrt{2}} \\ 2\sin^{-1}x & -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \dfrac{1}{\sqrt{2}} \le x \le 1 \end{cases}$



(g) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1} x & 0 \le x \le 1 \\ 2\pi - 2\cos^{-1} x & -1 \le x \le 0 \end{cases}$



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CHAPTER 18

Matrices

SPECIAL TYPE OF MATRICES

(*a*) **Row Matrix (Row vector):** $A = [a_{11}, a_{12}, \ldots a_{1n}]$ i.e. row matrix has exactly one row.

(*b*) **Column Matrix (Column vector):** $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ i.e. column matrix has exactly one column.

(*c*) **Zero or Null Matrix:** $(A = O_{m \times n})$, A $m \times n$ matrix whose all entries are zero.

(*d*) **Horizontal Matrix:** A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

(*e*) **Vertical Matrix:** A matrix of order $m \times n$ is a vertical matrix if $m > n$.

(*f*) **Square Matrix:** (Order n) if number of rows = number of column, then matrix is a square matrix.

NOTES

(*i*) The pair of elements a_{ij} and a_{ji} are called Conjugate Elements.

(*ii*) The elements $a_{11}, a_{22}, a_{33}, \ldots a_{mn}$ are called Diagonal Elements. the line long which the diagonal elements lie is called "Principal or Leading diagonal." The quantity Σa_{11} = trace of the matrix written as, $t_r(A)$.

(*g*) **Unit/Identity Matrix:** A square matrix, in which every non-diagonal element is zero and every diagonal element is 1, is called unit matrix or an identity matrix,

i.e. $a_{ij} = \begin{cases} 0, \text{ when } i \neq j \\ 1, \text{ when } i = j \end{cases}$

(*h*) **Upper Triangular Matrix:** A square matrix $A = [a_{ij}]_{n \times n}$ is called a upper triangular matrix, if $a_{ij} = 0, \forall\ i > j$.

(*i*) **Lower Triangular Matrix:** A square matrix $A = [a_{ij}]_{n \times n}$ is called a lower triangular matrix, if $a_{ij} = 0, \forall\ i < j$.

(*j*) **Submatrix:** A matrix which is obtained from a given matrix by deleting any number of rows or columns or both is called a submatrix of the given matrix.

(*k*) **Equal Matrices:** Two matrices A and B are said to be equal, if both having same order and corresponding elements of the matrices are equal.

(*l*) **Principal Diagonal of a Matrix:** In a square matrix, the diagonal from the first element of the first row to the last element of the last row is called the principal diagonal of a matrix.

e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 6 & 5 \\ 1 & 1 & 2 \end{bmatrix}$, the principal diagonal of A is 1, 6, 2.

(*m*) **Singular Matrix:** A square matrix A is said to be singular matrix, if determinant of A denoted by det (A) or $|A|$ is zero, i.e. $|A| = 0$, otherwise it is a non-singular matrix.

EQUALITY OF MATRICES

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if,

(*a*) both have the same order.

(*b*) $a_{ij} = b_{ij}$ for each pair of i & j.

ALGEBRA OF MATRICES

Addition: $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(*a*) **Addition of matrices is commutative:** $A + B = B + A$.

(*b*) **Matrix addition is associative:** $(A + B) + C = A + (B + C)$.

MULTIPLICATION OF A MATRIX BY A SCALAR

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

MULTIPLICATION OF MATRICES (Row by Column)

Let A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$ then the matrix multiplication AB is possible if and only if $n = p$.

Let $A_{m \times n} = [a_{ij}]$ and $B_{n \times p} = [b_{ij}]$, then order of AB is $m \times p$

and $\boxed{(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}}$

CHARACTERISTIC EQUATION

Let A be a square matrix. Then the polynomial $|A - xI|$ is called as characteristic polynomial of A & the equation $|A - xI| = 0$ is called characteristic equation of A.

PROPERTIES OF MATRIX MULTIPLICATION

(a) $AB = O \;\;\Rightarrow\;\; A = O$ or $B = O$ (in general)

Note: If A and B are two non-zero matrices such that $AB = O$, then A and B are called the divisors of zero. If A and B are two matrices such that

(i) $AB = BA$ then A and B are said to commute

(ii) $AB = -BA$ then A and B are said to anticommute

(b) Matrix Multiplication is Associative

If A, B & C are conformable for the product AB & BC, then $(AB)C = A(BC)$, $(AB)C = A(BC)$.

(c) Distributivity: $\left.\begin{matrix} A(B+C) = AB + AC \\ (A+B)C = AC + BC \end{matrix}\right]$ Provided A, B and C are conformable for respective products.

POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX

(a) $A^m A^n - A^{m+n}$

(b) $(A^m)^n = A^{mn} = (A^n)^m$

(c) $I^m = I \; m, n \in N$

ORTHOGONAL MATRIX

A square matrix is said to be orthogonal matrix if $AA^T = I$.

NOTES

(i) The determinant value of orthogonal matrix is either 1 or –1. Hence orthogonal matrix is always invertible.

(ii) $AA^T = I = A^TA$. Hence $A^{-1} = AT$.

SOME SQUARE MATRICES

(b) Idempotent Matrix: A square matrix is idempotent provided $A^2 = A$.

For idempotent matrix note the following:

(i) $A^n = A \ \forall \ n \geq 2, n \in N$.

(ii) determinant value of idempotent matrix is either 0 or 1.

(iii) If idempotent matrix is invertible then its inverse will be identity matrix i.e. I.

(b) Periodic Matrix: A square matrix which satisfies the relation $A^{k+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.

Note that period of an idempotent matrix is 1.

(c) Nilpotent Matrix: A square matrix is said to be nilpotent matrix of order m, $m \in N$, if $A^m = O$, $A^{m-1} \neq O$.

Note that a nilpotent matrix will not be invertible.

(d) Involutary Matrix: If $A^2 = I$, the matrix is said to be an involutary matrix.

Note that $A = A^{-1}$ for an involutary matrix.

(e) If A and B are square matrices of same order and $AB = BA$ then

$$(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + \ldots + {}^nC_nB^n.$$

TRANSPOSE OF A MATRIX : (Changing Rows & Columns)

Let A be any matrix of order $m \times n$. Then A^T or $A' = [a_{ij}]$ for $1 \leq i \leq n$ & $1 \leq j \leq m$ of order $n \times m$.

Properties of Transpose

If A^T & B^T denote the transpose of A and B

(*a*) $(A + B)^T = A^T + B^T$; note that A & B have the same order.

(*b*) $(A\,B)^T = B^T A^T$ (Reversal law) A & B are conformable for matrix product AB

(*c*) $(A^T)^T = A$

(*d*) $(kA)^T = kA^T$, where is a scalar.

General: $(A_1 \cdot A_2, \ldots A_n)^T = A^T{}_n \cdot \ldots A^T{}_2 \cdot A^T{}_1$ (reversal law for transpose)

SYMMETRIC & SKEW SYMMETRIC MATRIX

(*a*) **Symmetric matrix :** For symmetric matrix $A = A^T$.

Note: Maximum number of distinct entries in any symmetric matrix of order n is $\dfrac{n(n+1)}{2}$.

(*b*) **Skew symmetric matrix :** Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji}\ \forall\ i\ \&\ j$. Hence if A is skew symmetric, then $a_{ii} = -a_{ii} \quad \Rightarrow \quad a_{ii} = 0\ \forall\ i$.

Thus the diagonal elements of a skew square matrix are all zero, but not the converse.

For a skew symmetric matrix $A = -A^T$.

(*c*) **Properties of symmetric & skew symmetric matrix:**

(*i*) Let A be any square matrix then, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.

(*ii*) The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.

(*iii*) If A & B are symmetric matrices then,

1. $AB + BA$ is a symmetric matrix.
2. $AB - BA$ is a skew symmetric matrix.

(*iv*) Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}}$$

and $$A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$$

ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is $\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$. Then (adj A)

$$= \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}.$$

NOTES

If A be a square matrix of order n, then

(i) $A(\text{adj } A) = |A|\, I_n = (\text{adj } A) \, . \, A$

(ii) $|\text{ adj } A\,| = |A|^{n-1}$

(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{ adj}(\text{adj } A)\,| = |A|^{(n-1)^2}$

(v) $\text{adj } (AB) = (\text{adj } B)\, (\text{adj } A)$

(vi) $\text{adj } (KA) = K^{n-1}\, (\text{adj } A)$, where K is a scalar

INVERSE OF A MATRIX (Reciprocal Matrix)

A square matrix A said to be invertible (non singular if there exists a matrix B such that, $AB = I = BA$.

B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus

$$A^{-1} = B \quad \Leftrightarrow \quad AB = I = BA$$

We have, $\quad A \cdot (\text{adj } A) = |A|\, I_n$

$$A^{-1} \cdot A(\text{adj } A) = A^{-1}\, I_n\, |A|$$

$$I_n\, (\text{adj } A) = A^{-1}\, |A|\, I_n$$

$$\therefore \qquad A^{-1} = \frac{(\text{adj } A)}{|\,A\,|}$$

Note: The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Theorem: If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

NOTES

(*i*) If A be an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

(*ii*) If A is invertible,

(*a*) $(A^{-1})^{-1} = A$

(*b*) $(A^k)^{-1} = (A^{-1})^k = A^{-k}\,;\ k \in N$

SYSTEM OF EQUATION AND CRITERIA FOR CONSISTENCY

Gauss - Jordan Method

Example:

$$a_1x + b_1y = c_1z = d_1$$
$$a_2x + b_2y = c_2z = d_2$$
$$a_3x + b_3y = c_3z = d_3$$

$$\Rightarrow \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow \quad AX = B \quad \Rightarrow \quad A^{-1}AX = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}B = \frac{\text{Adj } A}{|A|} \cdot B$$

NOTES

(*i*) If $|A| \neq 0$, system is consistent having unique solution.

(*ii*) If $|A| \neq 0$ and $(\text{adj } A) \cdot B \neq O$ (Null matrix), system is consistent having unique non-trivial solution.

(*iii*) If $|A| \neq 0$ and $(\text{adj } A) \cdot B = O$ (Null matrix), system is consistent having trivial solution.

(*iv*) If $|A| = 0$, then

Matrix Method Fails

If $(\text{adj } A) \cdot B = O$ (null matrix) $\downarrow$ Consistent (infinite solutions)

If $(\text{adj } A) \cdot B \neq O$ $\downarrow$ Inconsistent (no solutions)

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CHAPTER 19

Determinants

PROPERTIES OF DETERMINANTS

(*i*) The value of the determinant remains unchanged, if rows are changed into columns and columns are changed into rows.

e.g. $$|A'| = |A|$$

(*ii*) If $A = [a_{ij}]_{n \times n}$, $n > 1$ and B be the matrix obtained from A by interchanging two of its rows or columns, then

$$\det(B) = -\det(A)$$

(*iii*) If two rows (or columns) of a square matrix A are proportional, then $|A| = 0$.

(*iv*) $|B| = k|A|$, where B is the matrix obtained from A, by multiplying one row (or column) of A by k.

(*v*) $|kA| = k^n |A|$, where A is a matrix of order $n \times n$.

(*vi*) If each element of a row (or column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

e.g. $$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$$

(*vii*) If the same multiple of the elements of any row (or column) of a determinant are added to the corresponding elements of any other row (or column), then the value of the new determinant remains unchanged, e.g. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+ka_{31} & a_{12}+ka_{32} & a_{13}\ \ ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(viii) If each element of a row (or column) of a determinant is zero, then its value is zero.

(ix) If any two rows (or columns) of a determinant are identical, then its value is zero.

(x) If r rows (or r columns) become identical, when a is substituted for x, then $(x - a)^{r-1}$ is a factor of given determinant.

IMPORTANT RESULTS ON DETERMINANTS

(i) $|AB| = |A||B|$, where A and B are square matrices of the same order.

(ii) $|A^n| = |A|^n$.

(iii) If A, B and C are square matrices of the same order such that i^{th} columns (or rows) of A is the sum of i^{th} columns (or rows) of B and C and all other columns (or rows) of A, B and C are identical, then $|A| = |B| + |C|$.

(iv) $|I_n| = 1$, where I_n is identity matrix of order n.

(v) $|O_n| = 0$, where O_n is a zero matrix of order n.

(vi) If $\Delta(x)$ has a 3rd order determinant having polynomials as its elements.

(a) If $\Delta(a)$ has 2 rows (or columns) proportional, then $(x - a)$ is a factor of $\Delta(x)$.

(b) If $\Delta(a)$ has 3 rows (or columns) proportional, then $(x - a)^2$ is a factor of $\Delta(x)$.

(vii) A square matrix A is **non-singular**, if $|A| \neq 0$ and **singular**, if $|A| = 0$.

(viii) Determinant of a skew-symmetric matrix of odd order is zero and of even order is a non-zero perfect square.

(ix) In general, $|B + C| \neq |B| + |C|$.

(x) Determinant of a diagonal matrix = Product of its diagonal elements

(xi) If A is a non-singular matrix, then $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$.

(*xii*) Determinant of a orthogonal matrix = 1 or – 1.

(*xiii*) Determinant of a hermitian matrix is purely real.

(*xiv*) If A and B are non-zero matrices and $AB = O$, then it implies $|A| = O$ and $|B| = O$.

Minors and Cofactors

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then the **minor** M_{ij} of the element a_{ij} is the determinant obtained by deleting the i^{th} row and j^{th} column,

i.e. M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The **cofactor** of the element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Properties of Minors and Cofactors

(*i*) The sum of the products of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero,

i.e. if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$ and so on.

(*ii*) The sum of the product of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of the same row (or column) in Δ,

i.e. if $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $|A| = \Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$.

(*iii*) In general, if $|A| = \Delta$, then $|A| = \sum_{i=1}^{n} a_{ij}\, C_{ij}$ and $(\text{adj } A) = \Delta^{n-1}$, where A is a matrix of order $n \times n$.

Applications of Determinants in Geometry

Let the three points in a plane be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

(*i*) Area of $\Delta ABC = \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

(*ii*) If the given points are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

(*iii*) Let two points are $A(x_1, y_1)$, $B(x_2, y_2)$ and $P(x, y)$ be a point on the line joining points A and B, then the equation of line is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$

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CHAPTER 20

Limits, Continuity and Differentiability

1. Limit of a function f(x) is said to exist as x → a when,

$$\lim_{h\to 0^+} f(a-h) = \lim_{h\to 0^+} f(a+h) = \text{some finite value } M.$$

(Left hand limit) (Right hand limit)

2. Indeterminate Forms

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, \infty^{\circ}, 0^{\circ}, \text{ and } 1^{\infty}.$$

3. Standard Limits

$$\underset{x\to 0}{\text{Limit}} \frac{\sin x}{x} = \underset{x\to 0}{\text{Limit}} \frac{\tan x}{x} = \underset{x\to 0}{\text{Limit}} \frac{\tan^{-1} x}{x} = \underset{x\to 0}{\text{Limit}} \frac{\sin^{-1} x}{x}$$

$$= \underset{x\to 0}{\text{Limit}} \frac{e^x - 1}{x} = \underset{x\to 0}{\text{Limit}} \frac{\ell n(1+x)}{x} = 1,$$

$$\underset{x\to 0}{\text{Limit}} (1+x)^{1/x} = \underset{x\to \infty}{\text{Limit}} \left(1+\frac{1}{x}\right)^x = e, \ \underset{x\to 0}{\text{Limit}} \frac{a^x - 1}{x} = \log_e a, \ a > 0,$$

$$\underset{x\to a}{\text{Limit}} \frac{x^n - a^n}{x-a} = na^{n-1}.$$

FUNDAMENTAL THEOREMS ON LIMITS

Let $\lim_{x\to a} f(x) = l$ & $\lim_{x\to a} g(x) = m$. If l and m exists finitely then:

(a) Sum rule : $\lim_{x\to a}[f(x)+g(x)] = l+m$

(b) Difference rule : $\lim_{x\to a}[f(x)-g(x)] = l-m$

(c) Product rule : $\lim_{x\to a} f(x).g(x) = l.m$

(d) Quotient rule : $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{l}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\lim\limits_{x\to a} kf(x)=k\lim\limits_{x\to a} f(x)$; where k is constant.

(f) Power rule : If m and n are integers, then $\lim\limits_{x\to a}\left[f(x)\right]^{m/n}=l^{m/n}$, provided $l^{m/n}$ is a real number.

(g) $\lim\limits_{x\to a} f[g(x)]=f\left(\lim\limits_{x\to a} g(x)\right)=f(m)$; provided $f(x)$ is continuous at $x=m$.

4. Limits Using Expansion

(i) $a^x=1+\dfrac{xlna}{1!}+\dfrac{x^2ln^2a}{2!}+\dfrac{x^3ln^3a}{3!}+\ldots\ldots., a>0$

(ii) $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots\ldots.$

(iii) $ln\ (1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots\ldots.$, for $-1<x\leq 1$

(iv) $\sin\ x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\ldots\ldots.$

(v) $\cos\ x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\ldots\ldots.$

(vi) $\tan\ x=x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\ldots\ldots$

(vii) $\tan^{-1}\ x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots\ldots$

(viii) $\sin^{-1}\ x=x+\dfrac{1^2}{3!}x^3+\dfrac{1^2.3^2}{5!}x^5+\dfrac{1^2.3^2.5^2}{7!}x^7+\ldots\ldots$

(x) for $|x|<1, n\in R, (1+x)^n=1+nx+\dfrac{n(n-1)}{1.2}$

$x^2+\dfrac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\ldots.\infty$

5. Limits of form 1^∞, 0^0, ∞^0

Also for $(1)^\infty$ type of problems we can use following rules.

$\lim\limits_{x\to 0}(1+x)^{1/x}=e, \lim\limits_{x\to a}[f(x)]^{g(x)}$, where $f(x)\to 1$; $g(x)\to\infty$ as $x\to a$

$\lim\limits_{x\to a}[f(x)]^{g(x)}=e^{\lim\limits_{x\to a}\langle f(x)-1\rangle g(x)}$

6. Sandwich Theorem or Squeeze Play Theorem

If $f(x) \le g(x) \le h(x) \forall x$ and $\lim\limits_{x \to a} f(x) = l = \lim\limits_{x \to a} h(x)$, then $\lim\limits_{x \to a} g(x) = \ell$

PROPERTIES OF CONTINUOUS FUNCTIONS

Here we present two extremely useful properties of continuous functions; Let $y = f(x)$ be a continuous function $\forall\ x \in [a, b]$, then following results hold true.

(*i*) f is bounded between a and b. This simply means that we can find real numbers m_1 and m_2 such $m_1 \le f(x) \le m_2\ \forall\ x \in [a, b]$.

(*ii*) Every value between $f(a)$ and $f(b)$ will be assumed by the function atleast once. This property is called intermediate value theorem of continuous function.

In particular if $f(a) \cdot f(b) < 0$, then $f(x)$ will become zero atleast once in (a, b). It also means that if $f(a)$ and $f(b)$ have opposite signs then the equation $f(x) = 0$ will have atleast one real root in (a, b).

TYPES OF DISCONTINUITIES

Type-1 : (Removable type of discontinuities)

(*a*) **Missing point discontinuity :** Where $\lim\limits_{x \to a} f(x)$ exists finitely but $f(a)$ is not defined.

(*b*) **Isolated point discontinuity :** Where $\lim\limits_{x \to a} f(x)$ exists & $f(a)$ also exists but; $\lim\limits_{x \to a} f(x) \ne f(a)$.

Type-2 : (Non-Removable type of discontinuities)

(*a*) **Finite type discontinuity :** In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

(*b*) **Infinite type discontinuity :** In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.

(*c*) **Oscillatory type discontinuity :** Limits oscillate between two finite quantities.

Derivability of function at a point

If $f'(a^+) = f'(a^-) =$ finite quantity, then $f(x)$ is said to be **derivable or differentiable at $x = a$.** In such case $f'(a^+) = f'(a^-) = f'(a)$ and it is called derivative or differential coefficient of $f(x)$ at $x = a$.

NOTE

(i) All polynomial, trigonometric, inverse trigonometric, logarithmic and exponential function are continuous and differentiable in their domains, except at end points.

(ii) If $f(x)$ and $g(x)$ are derivable at $x = a$ then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).\ g(x)$ will also be derivable at $x = a$ and if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be derivable at $x = a$.

In short, for a function 'f':

Differentiable $\Rightarrow$ **Continuous;**

Not Differentiable $\Rightarrow$ **Not Continuous**

But Not Continuous $\Rightarrow$ **Not Differentiable**

Continuous $\Rightarrow$ **May or may not be Differentiable**

DERIVABILITY OVER AN INTERVAL

(a) $f(x)$ is said to be derivable over an open interval (a, b) if it is derivable at each and every point of the open interval (a, b).

(b) $f(x)$ is said to be derivable over the closed interval (a, b) if :

(i) $f(x)$ is derivable in (a, b) and

(ii) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist.

NOTE

(i) If $f(x)$ is differentiable at $x = a$ and$g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x).g(x)$ can still be differentiable at $x = a$.

(ii) If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function; $F(x) = f(x).g(x)$ can still be differentiable at $x = a$.

(iii) If $f(x)$ & $g(x)$ both are non-derivable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function.

(iv) If $f(x)$ is derivable at $x = a$ $\not\Rightarrow$ $f'(x)$ is continuous at $x = a$.

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CHAPTER 21

Method of Differentiation

DIFFERENTIATION OF SOME ELEMENTARY FUNCTIONS

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(a^x) = a^x \ell n\, a$

3. $\frac{d}{dx}(\ell n\,|x|) = \frac{1}{x}$

4. $\frac{d}{dx}(\log_a x) = \frac{1}{x\, \ell n\, a}$

5. $\frac{d}{dx}(\sin x) = \cos x$

6. $\frac{d}{dx}(\cos x) = -\sin x$

7. $\frac{d}{dx}(\sec x) = \sec x \tan x$

8. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

9. $\frac{d}{dx}(\tan x) = \sec^2 x$

10. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

BASIC THEOREMS

1. $\frac{d}{dx}(f \pm g)(x) = f'(x) \pm g'(x)$

2. $\frac{d}{dx}(k\, f(x)) = k\frac{d}{dx} f(x)$

3. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x)$

4. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$

5. $\dfrac{d}{dx}\big(f(g(x))\big)=f'(g(x))g'(x)$

DERIVATIVE OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d\sin^{-1}x}{dx}=\frac{1}{\sqrt{1-x^2}},\ \frac{d\cos^{-1}x}{dx}=-\frac{1}{\sqrt{1-x^2}},\text{ for }-1<x<1.$$

$$\frac{d\tan^{-1}x}{dx}=\frac{1}{1+x^2},\ \frac{d\cot^{-1}x}{dx}=-\frac{1}{1+x^2}\quad (x\in R)$$

$$\frac{d\sec^{-1}x}{dx}=\frac{1}{|x|\sqrt{x^2-1}},\ \frac{d\,\text{cosec}^{-1}x}{dx}=-\frac{1}{|x|\sqrt{x^2-1}},\text{ for }x\in(-\infty,-1)\cup(1,\infty)$$

DIFFERENTIATION USING SUBSTITUTION

Following substitutions are normally used to simplify these expression.

(i) $\sqrt{x^2+a^2}$ by substituting $x=a\tan\theta$, where $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$

(ii) $\sqrt{a^2-x^2}$ by substituting $x=a\sin\theta$, where $-\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}$

(iii) $\sqrt{x^2-a^2}$ by substituting $x=a\sec\theta$, where $\theta\in[0,\pi]$, $\theta\ne\dfrac{\pi}{2}$

(iv) $\sqrt{\dfrac{x+a}{a-x}}$ by substituting $x=a\cos\theta$, where $\theta\in[0,\pi]$.

PARAMETRIC DIFFERENTIATION

If $y=f(\theta)$ & $x=g(\theta)$ where θ is a parameter, then $\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}$.

DERIVATIVE OF ONE FUNCTION WITH RESPECT TO ANOTHER

Let $y = f(x)$; $z = g(x)$ then $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{f'(x)}{g'(x)}$.

❖ If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where $f, g, h, l, m, n, u, v, w$ are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g'(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

L' HOPITAL'S RULE

Applicable while calculating limits of indeterminate forms of the type $\dfrac{0}{0}, \dfrac{\infty}{\infty}$. If the function $f(x)$ and $g(x)$ are differentiable in certain neighbourhood of the point a, except, may be, at the point a itself, and $g'(x) \neq 0$, and if

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \text{ or } \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty,$$

then $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

provided the limit $\displaystyle \lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists **(L' Hopital's rule)**, The point 'a' may be either finite or improper $+\infty$ or $-\infty$.

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CHAPTER 22

Application of Derivatives

DERIVATIVE TEST FOR INCREASING AND DECREASING FUNCTIONS AT a POINT

1. If $f'(a) > 0$ then $f(x)$ is increasing at $x = a$.
2. If $f'(a) < 0$ then $f(x)$ is decreasing at $x = a$.
3. If $f'(a) = 0$ then examine the sign of $f'(a^+)$ and $f'(a^-)$.
   - (*a*) If $f'(a^+) > 0$ and $f'(a^-) > 0$ then increasing
   - (*b*) If $f'(a^+) < 0$ and $f'(a^-) < 0$ then decreasing
   - (*c*) Otherwise neither increasing nor decreasing

NOTES

Above rule is applicable only for functions that are differentiable at $x = a$.

SPECIAL POINTS

(*a*) **Critical points:** The points of domain for which $f'(x)$ is equal to zero or doesn't exist are called critical points.

(*b*) **Stationary points:** The stationary points are the points of domain where $f'(x) = 0$.

Every stationary point is a critical point.

SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE AND POINT OF INFLECTION

If $f''(x) > 0 \ \forall \ x \in (a, b)$ then graph of $f(x)$ is concave upward in (a, b). Similarly if $f''(x) < 0 \ \forall \ x \in (a, b)$ then graph of $f(x)$ is concave downward in (a, b).

Point of inflection

For finding point of inflection of any function, compute the solutions of $\frac{d^2y}{dx^2} = 0$ or does not exist and check the sign of y'' about these points.

EQUATION OF TANGENT AND NORMAL

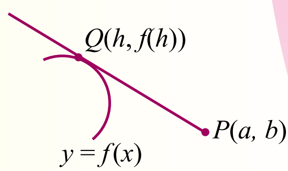
Tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

And normal at (x_1, y_1) is $(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

TANGENT FROM AN EXTERNAL POINT

Given a point $P(a, b)$ which does not lie on the curve $y = f(x)$, then the equation of possible tangents to the passing through (a, b) can be found by solving for the point of contact Q.

$$f'(h) = \frac{f(h) - b}{h - a}$$



And equation of tangent is $y - b = \frac{f(h) - b}{h - a}(x - a)$

LENGTH OF TANGENT, NORMAL, SUBTANGENT, SUBNORMAL

(i) $PT = |k| \sqrt{1 + \frac{1}{m^2}}$ = Length of Tangent (*ii*) $PN = |k|\ \sqrt{1+m^2}$ = Length of Normal

(*iii*) $TM = \left|\dfrac{k}{m}\right|$ = Length of subtangent

(*iv*) $MN = |\text{km}|$ = Length of subnormal.

ANGLE BETWEEN THE CURVES

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

SHORTEST DISTANCE BETWEEN TWO CURVES

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

ROLLE'S THEOREM

If *a* function *f* defined on $[a, b]$ is

(*i*) Continuous on $[a, b]$

(*ii*) derivable on (a, b) and

(*iii*) $f(a) = f(b)$.

then there exists atleast one c $(a < c < b)$ such that $f'(c) = 0$.

LAGRANGE'S MEAN VALUE THEOREM (LMVT)

If *a* function *f* defined on $[a, b]$ is

(*i*) continuous on $[a, b]$ is

(*ii*) derivable on (a, b)

(*iii*) $f(a) = f(b)$,

then there exists at least one real numbers between a and b $(a < c < b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

USEFUL FORMULAE OF MENSURATION TO REMEMBER

1. Volume of a cuboid = ℓbh.
2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.
3. Volume of cube = a^3.
4. Surface area of cube = $6a^2$.
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.
6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height).
7. Curved surface area of a cylinder = $2\pi rh$.
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
9. Volume of a sphere = $\frac{4}{3}\pi r^3$.
10. Surface area of a sphere = $4\pi r^2$.
11. Area of a circular sector = $\frac{1}{2}r^2\theta$, when θ is in radians.
12. Volume of a prism = (area of the base) × (height).
13. Lateral surface area of a prism = (perimeter of the base) × (height).
14. Total surface area of a prism = (lateral surface area) + 2 (area of the base).

    (Note that lateral surfaces of a prism are all rectangle.)
15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).
16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).

    (Note that slant surfaces of a pyramid are triangles).

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CHAPTER 23

Indefinite Integration

1. If f & g are functions of x such that g′(x) = f(x) then,

$\int f(x)dx = g(x)+c \Leftrightarrow \frac{d}{dx}\{g(x)+c\} = f(x)$, where c is called the **constant of integration.**

2. Standard Formula

(i) $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n\,|ax+b| + c$

(iii) $\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + c$

(iv) $\int a^{px+q} dx = \frac{1}{p}\frac{a^{px+q}}{\ell na} + c; a > 0$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)dx = \frac{1}{a}\ell n \sec|(ax+b)| + c$

(viii) $\int \cot(ax+b)dx = \frac{1}{a}\ell n\,|\sin(ax+b)| + c$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+c$

(x) $\int \operatorname{cosec}^2 (ax+b)dx = -\frac{1}{a}\cot(ax+b)+c$

(xiii) $\int \sec x\, dx = \ell n\,|(\sec x+\tan x)|+c$ OR $\ell n\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+c$

(xiv) $\int \operatorname{cosec} x\, dx = \ell n\,|(\operatorname{cosec} x-\cot x)|+c$ OR $\ell n\left|\tan\frac{x}{2}\right|+c$

OR

$-\ell n\,|(\operatorname{cosec} x+\cot x)|+c$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$

(xvi) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

(xvii) $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left|x+\sqrt{x^2+a^2}\right|+c$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left|x+\sqrt{x^2-a^2}\right|+c$

(xx) $\int \frac{dx}{(a^2-x^2)} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right|+c$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right|+c$

(xxii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

(xxiii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ell n\left|\frac{x+\sqrt{x^2+a^2}}{a}\right|+c$

(xxiv) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ell n\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c$

3. Integration by substitutions

If we substitute $f(x) = t$, then $f'(x)\,dx = dt$

4. Integration by part

$$\int\left(f(x)g(x)dx = f(x)\right)\int(g(x))dx - \int\left(\frac{d}{dx}(f(x))\int(g(x))dx\right)dx$$

5. Integration of type

$$\int\frac{dx}{ax^2+bx+c}, \int\frac{dx}{\sqrt{ax^2+bx+c}}, \int\sqrt{ax^2+bx+c}\,bx$$

Make the substitution $x+\frac{b}{2a}=t$

6. Integration of type

$$\int\frac{px+q}{ax^2+bx+c}dx, \int\frac{px+q}{\sqrt{ax^2+bx+c}}dx, \int(px+q)\sqrt{ax^2+bx+c}\,dx$$

Make the substitution $x+\frac{b}{2a}$, then split the integral as some of two integrals one containing the linear term and the other containing constant term.

7. Integration of trigonometric functions

(*i*) $\int\frac{dx}{a+b\sin^2 x}$ OR, $\int\frac{dx}{a+b\cos^2 x}$ OR,

$\int\frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$, put $\tan x = t$

(*ii*) $\int\frac{dx}{a+b\sin x}$ OR $\int\frac{dx}{a+b\cos x}$ OR

$\int\frac{dx}{a+b\sin x+c\cos x}$, put $\tan\frac{x}{2}=t$.

(*iii*) $\int\frac{a.\cos x+b.\sin x+c}{\ell.\cos x+m.\sin x+n}dx$.Express

Nr $\equiv A(Dr)+B\frac{d}{dx}(Dr)+c$ & proceed.

8. Integration of type

$\int\frac{x^2\pm1}{x^4+Kx^2+1}dx$ where K is any constant.

Divide Nr & Dr by x^2 & put $x\mp\frac{1}{x}=t$

9. Integration of type

$$\int \frac{dx}{(ax+b)\sqrt{px+q}} \; OR \; \int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}} \text{ ; put } px+q=t^2.$$

10. Integration of type

$$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}, \text{ put } ax+b=\frac{1}{t};$$

$$\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}} \text{ put } x=\frac{1}{t}$$

Some standard substitution

1. $\int f(x)^n f'(x)dx$ OR $\int \frac{f'(x)}{[f(x)]^n}dx$ put $f(x) = t$ & proceed.

2. $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,dx$

   Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

3. $\int \frac{(px+q)dx}{ax^2+bx+c}, \int \frac{(px+q)}{\sqrt{ax^2+bx+c}}dx$

   Express $px + q = A$ (differential coefficient of denominator) $+ B$.

4. $\int e^x[f(x)+f'(x)]dx = e^x.f(x)+c$

5. $\int [f(x)+xf'(x)]dx = xf(x)+c$

6. $\int \frac{dx}{x(x^n+1)}, n \in N,$ take x^n common & put $1 + x^{-n} = t$.

7. $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}, n \in N,$ take x^n common & put $1 + x^{-n} = t^n$.

8. $\int \frac{dx}{x^n(1+x^n)^{1/n}},$ take x^n common and put $1 + x^{-n} = t$.

9. $\int \sqrt{\frac{x-\alpha}{\beta-x}}dx$ OR $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha \cos^2\theta + \beta \sin^2\theta$

   $\int \sqrt{\frac{x-\alpha}{x-\beta}}dx$ OR $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha \sec^2\theta - \beta \tan^2\theta$

   $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x - 2 = t^2$ or $x - \beta = t^2$.

CHAPTER 24

Definite Integration

1. (*a*) The Fundamental Theorem of Calculus Part 1:

If f is continuous on $[a, b]$, **then the function** g **defined by**

$$g(x) = \int_a^x f(t)dt \quad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

(*b*) The Fundamental theorem of Calculus, Part 2:

If is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f, that is, a function such that $F' = f$.

Note: If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b).

2. A definite integral is denoted by $\int_a^b f(x)dx$ which represent the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis.

ex. $\int_0^{2\pi} \sin x\, dx = 0$.

PROPERTIES OF DEFINITE INTEGRAL

1. $\int_a^b f(x)\,dx = \int_a^b f(t)dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

4. $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x)+f(-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(-x)=f(x) \\ 0, & f(-x)=-f(x) \end{cases}$

5. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x))dx$

6. $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

7. $\int_0^{2a} f(x)dx = \int_0^a (f(x)+f(2a-x))\,dx = \begin{cases} 2\int_0^a f(x)dx, & f(2a-x)=f(x) \\ 0, & f(2a-x)=-f(x) \end{cases}$

8. If $f(x)$ is a periodic function with period T, then

$\int_0^{nT} f(x)dx = n\int_0^T f(x)\,dx, n \in Z,$ $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx, n \in Z, a \in R$

$\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx, m,n \in Z,$ $\int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)dx, n \in Z, a \in R$

$\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx, n \in Z, a,b \in R$

9. If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then $\int_a^b \Psi(x)dx \le \int_a^b f(x)dx \le \int_a^b \phi(x)dx$

Leibnitz Theorem: If $F(x) = \int_{g(x)}^{h(x)} f(t)dt,$

then $\dfrac{dF(x)}{dx} = h'(x)f(h(x)) - g'(x)f(g(x))$

WALLI'S FORMULA

(a) $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)....(1 \text{ or } 2)}{n(n-2).....(1 \text{ or } 2)} K$$

where $$K = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

(b) $$\int_0^{\pi/2} \sin^n x.\cos^m x\,dx$$

$$= \frac{[(n-1)(n-3)(n-5)....1 \text{ or } 2][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2} K$$

where $$K = \begin{cases} \dfrac{\pi}{2} & \text{if both } m \text{ and } n \text{ are even } (m, n \in N) \\ 1 & \text{otherwise} \end{cases}$$

DEFINITE INTEGRAL AS LIMIT OF A SUM

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+\overline{n-1}h)]$$

$$\Rightarrow \quad \lim_{h\to\infty} h\sum_{r=0}^{n-1} f(a+rh) = \int_0^1 f(x)dx \text{ where } b-a = nh$$

If $a = 0$ & $b = 1$ then, $\lim\limits_{n\to\infty} h\sum\limits_{r=0}^{n-1} f(rh) = \int_0^1 f(x)dx$; where $nh = 1$

OR $$\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx.$$

ESTIMATION OF DEFINITE INTEGRAL

(a) If $f(x)$ is continuous in $[a, b]$ and it's range in this interval is $[m, M]$, them $m\,(b-a) \le \int_a^b f(x)dx \le M(b-a)$

(b) If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b \phi(x)dx$

(c) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx.$

(d) If $f(x) \ge 0$ on the interval $[a, b]$, then $\int_a^b f(x)dx \ge 0.$

(e) $f(x)$ and $g(x)$ are two continuous function on $[a, b]$ then

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \le \sqrt{\int_a^b f^2(x)dx \int_a^b g^2(x)dx}$$

SOME STANDARD RESULTS

(a) $\int_0^{\pi/2} \log \sin x\, dx = -\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log \cos x\, dx$

(b) $\int_a^b \frac{|x|}{x}\, dx = |b| - |a|.$

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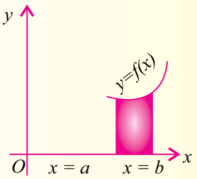
CHAPTER 25

Area Under The Curves

1. The area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ & $x = b$ is given by,

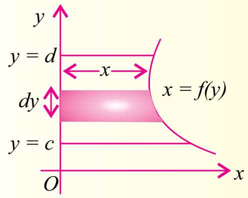
$$A = \int_a^b f(x)dx = \int_a^b y\,dx$$



2. If the area is below the x-axis then A is negative. The convention is to consider the magnitude only *i.e.* $A = \left|\int_a^b y\,dx\right|$ in this case.

3. The area bounded by the curve $x = f(y)$, y-axis & abscissa $y = c$, $y = d$ is given by,

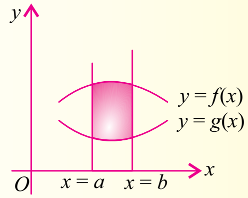
$$\text{Area} = \int_c^d x dy = \int_c^d f(y)dy$$



4. Area between the curves $y = f(x)$ & $y = g(x)$ between the ordinates $x = a$ & $x = b$ is given by,

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$



5. Average value of a function $y = f(x)$ *w.r.t.* x over an interval $a \le x \le b$ is defined as: $y_{av} = \dfrac{1}{b-a}\int_a^b f(x)dx.$

6. Curve Tracing:

The following outline procedure is to be applied in Sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(*a*) Symmetry: The symmetry of the curve is judged as follows:

(*i*) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

(*ii*) If all the powers of x are even, the curve is symmetrical about the axis of y.

(*iii*) If powers of x & y both are even, the curve is symmetrical about the axis of x as well as y.

(*iv*) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

(*v*) If on interchanging the signs of x & y both the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(*b*) Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

(*c*) Find the points where the curve crosses the x-axis & also the y-axis.

(*d*) Examine if possible the intervals when $f(x)$ is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

7. Useful Results:

(*a*) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is $\pi\, ab$.

(*b*) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is $16ab/3$.

(*c*) Area included between the parabola $y^2 = 4\,ax$ & the line $y = mx$ is $8\,a^2/3\,m^3$.

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CHAPTER 26

Differential Equations

ORDER OF A DIFFERENTIAL EQUATION

The order of highest order derivative appearing in a differential equation is called the order of the differential equation.

DEGREE OF A DIFFERENTIAL EQUATION

The degree of an algebraic differential equation is the degree of the derivative (or differential) of the highest order in the equation, after the equation is freed from radicals and fractions in its derivatives.

Variable Separable Differentiable Equations

A differential equation of the form $f(x) + g(y)\dfrac{dy}{dx} = 0$

EQUATIONS REDUCIBLE TO SEPARABLE FORM

$\dfrac{dy}{dx} = f\,(ax + by + c)$ can be reduced to varible separable form by substitution $ax + by + c = t$. The reduced variable separable form is:

$$\frac{dt}{bf\,(t) + a} = dx.$$

HOMOGENEOUS DIFFERENTIAL EQUATION

$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ where $f(x, y)$ and $g(x, y)$ are both homogeneous function of same degree in x and y.

Substitute $y = vx$ and so $\frac{dy}{dx} = v + x\frac{dv}{dx}$

EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM

Consider a differential equation of the form:

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \text{ where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Put $x = x + h$

$y = Y + k$

LINEAR EQUATION

An equation of the form $\frac{dy}{dx} + Py = Q$

Multiply both sides of the equation by $e^{\int Pdx}$.

$$\therefore \quad \frac{dy}{dx}e^{\int Pdx} + Pye^{\int Pdx} = Qe^{\int Pdx}$$

Bernoulli's Equation

An equation of the form $\frac{dy}{dx} + Py = Qy^n$,

Putting $y^{-n+1} = v$

$$\Rightarrow \quad \frac{dv}{dx} + (1 - n)\, P \cdot\, y = (1 - n)Q.$$

NOTES

Following exact differentials must be remembered:

(i) $xdy + ydx = d(xy)$

(ii) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{xdy + ydx}{xy} = d(\ell nxy)$

(v) $\dfrac{dx + dy}{x + y} = d(\ell n(x + y))$

(vi) $\dfrac{xdy - ydx}{xy} = d\left(\ell n\dfrac{y}{x}\right)$

(vii) $\dfrac{ydx - xdy}{xy} = d\left(\ell n\dfrac{x}{y}\right)$

(viii) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(ix) $\dfrac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$

(x) $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\ell n\sqrt{x^2 + y^2}\right]$

(xi) $d\left(-\dfrac{1}{xy}\right) = \dfrac{xdy + ydx}{x^2y^2}$

(xii) $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^x dx - e^x dy}{y^2}$

(xiii) $d\left(\dfrac{e^y}{x}\right) = \dfrac{xe^y dy - e^y dx}{x^2}$

ORTHOGONAL TRAJECTORY

Any curve which cuts every member of a given family of curves at right angle is called an *orthogonal trajectory* of the family. *For example*, each straight line $y = mx$ passing through the origin, is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$.

Procedure for finding the orthogonal trajectory

(i) Let $f(x, y, c) = 0$ be the equation, where c is an arbitrary parameter.

(ii) Differentiate the given equation w.r.t. x and then eliminate c.

(iii) Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in the equation obtained in (*ii*).

(iv) Solve the differential equation in (*iii*).

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CHAPTER 27

Vectors

POSITION VECTOR OF A POINT

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two A and B, then, $\overrightarrow{AB} = \vec{b} - \vec{a} = pv$ of $B - pv$ of A.

Distance Formula

Distance between the two points $A(\vec{a})$ and $B(\vec{b})$ is $AB = |\vec{a} - \vec{b}|$

Section Formula

$$\vec{r} = \frac{n\vec{a} + m\vec{b}}{m+n} \qquad \text{Mid point of } AB = \frac{\vec{a}+\vec{b}}{2}.$$

SCALAR PRODUCT OF TWO VECTORS

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, where $|\vec{a}|, |\vec{b}|$ are magnitude of $\vec{a}$ and $\vec{b}$ respectively and θ is angle between $\vec{a}$ and $\vec{b}$.

1. $\hat{i}\cdot\hat{i} = \hat{j}\,\hat{j} = \hat{k}\,\hat{k} = 1$; $\hat{i}\cdot\hat{j} = \hat{j}\,\hat{k} = \hat{k}\,\hat{i} = 0$; i projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.
2. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} - b_3\hat{k}$ then $\vec{a}\,.\,\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

3. The angle ϕ between $\vec{a}$ and $\vec{b}$ is given by $\cos\phi = \frac{\vec{a}\,\vec{b}}{|\vec{a}|\,|\vec{b}|}, 0 \le \phi \le \pi$.

4. $\vec{a}\,.\,\vec{b} = 0 \quad \Leftrightarrow \quad \vec{a} \perp \vec{b} \qquad (\vec{a} \ne \vec{b} \ne 0)$

VECTOR PRODUCT OF TWO VECTORS

1. If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$ $\sin\theta\,\vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}, \vec{b}$ & $\vec{n}$ forms a right handed screw system.

2. Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.

3. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}; \hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, k \times i = \hat{j}.$

4. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

5. $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) ($\vec{a} \ne 0$, $\vec{b} \ne 0$) i.e. $\vec{a} = k\vec{b}$, where K is a scalar.

6. Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

❖ If $\vec{a}$, $\vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B and C then the vector area of triangle $ABC = \frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$. The points A, B and C are collinear If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$.

❖ Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$.

❖ Lagrange's identity : $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$.

SCALAR TRIPLE PRODUCT

- The scalar product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as: $\vec{a} \times \vec{b} \cdot \vec{c} = |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\phi$.

- Volume of tetrahedron v = $[\vec{a}\ \vec{b}\ \vec{c}\,]$.

- In a scalar triple product the position of dot & cross can be interchanged i.e. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ *OR* $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$.

- $\vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$.

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} - b_3\hat{k}$ & $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$$[\vec{a}\ \vec{b}\ \vec{c}\,] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

In general, If

$\vec{a} = a_1\vec{\ell} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{\ell} + b_2\vec{m} + b_3\vec{n}$ & $\vec{c} = c_1\vec{\ell} + c_2\vec{m} + c_3\vec{n}$ then

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{\ell}\ \vec{m}\ \vec{n}];$$ where $\vec{\ell}, \vec{m}$ & $\vec{n}$ are non coplanar vectors.

- If $\vec{a}, \vec{b}\ \vec{c}$ are coplannar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}\,] = 0$.

- Volume of tetrahedron *OABC* with *O* as origin and $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ be the vertices $= \left|\frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]\right|$.

- The position vector of the centroid of a tetrahedron if the *pv's* of its vertices are $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}[\vec{a} + \vec{b} + \vec{c} + \vec{d}]$.

VECTOR TRIPLE PRODUCT

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\times\vec{c})\vec{c},\ (\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$

❖ $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$, in general

RECIPROCAL SYSTEM OF VECTORS

If $\vec{a}$, $\vec{b}$, $\vec{c}$ & m $\vec{a'}, \vec{b'}, \vec{c'}$ are two sets of non coplanar vectors such that $\vec{a}, \vec{a'} = \vec{b}\,\vec{b'} = \vec{c}\,\vec{c'} = 1$ then the two systems are called Reciprocal System of Vectors, where $\vec{a'} = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{b'} = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{c'} = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$.

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CHAPTER 28

Three Dimensional Geometry

1. **Vector representation of a point :** Position vector of point

$$P(x, y, z) \text{ is } x\hat{i} + y\hat{j} + z\hat{k}.$$

2. **Distance formula:** $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$, $AB = |\overrightarrow{OB} - \overrightarrow{OA}|$

3. **Distance of P from coordinate axes :**

$$PA = \sqrt{y^2 + z^2}, PB = \sqrt{z^2 + x^2}, PC = \sqrt{x^2 + y^2}$$

4. **Section Formula :** $x = \dfrac{mx_2 + nx_1}{m+n}, y = \dfrac{my_2 + n_1}{m+n}, z = \dfrac{mz_2 + nz_1}{m+n}$

   Mid point : $x = \dfrac{x_1 + x_2}{2}, y = \dfrac{y_1 + y_2}{2}, z = \dfrac{z_1 + z_2}{2}$

5. **Direction Cosines and Direction Ratios**

   (*i*) **Direction cosines :** Let α, β, γ be angles which a directed line makes with the positive directions of the axes of *x*, *y* and *z* respectively, then cos α, cosβ, cos γ are called the direction cosines of the line. The direction cosines are usually denoted by (*l*, *m*, *n*). Thus *l* = cos α, *m* = cos β, *n* = cos γ.

   (*ii*) If *l*, *m*, *n* be the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$.

   (*iii*) **Direction ratios:** Let *a*, *b*, *c* be proportional to the direction cosines *l*, m, n then *a*, *b*, *c* are called the direction ratios.

   (*iv*) If *l*, *m*, *n* be the direction cosines and *a*, *b*, *c* be the direction ratios of a vector, then

$$\ell = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

(*v*) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}$, $m = \frac{y_2 - y_1}{|PQ|}$ and $n = \frac{z_2 - z_1}{|PQ|}$.

6. Angle between Two Line Segments :

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|.$$

The line will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$, parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

7. Projection of a line segment on a line : If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then the projection of PQ on a line having direction cosines l, m, n is $\left|l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)\right|$.

8. Equation of A Plane : General form : $ax + by + cz + d = 0$, where a, b, c are not all zero, $a, b, c, d \varepsilon R$.

(*i*) Normal form : $lx + my + nz = p$

(*ii*) Plane through the point (x_1, y_1, z_1) : $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

(*iii*) Intercept Form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(*iv*) Vector form: $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ or $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

(*v*) Any plane parallel to the given plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.

Distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $= \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$.

(*vi*) **Equation of a plane passing through a given point & parallel to the given vectors:** $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.

or $\vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$ (non parametric form)

9. A plane & A Point

(*i*) Distance of the point (x', y', z') from the plane $ax + by + cz + d = 0$ is given by $\dfrac{ax' + by' + cz' + d}{\sqrt{a^2 + b^2 + c^2}}$.

(*ii*) Length of the perpendicular from a point $(\vec{a})$ to plane $\vec{r}.\vec{n} = d$ is given by $p = \dfrac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}$.

(*iii*) Foot (x', y', z') of perpendicular drawn from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is given by $\dfrac{x' - x_1}{a} = \dfrac{y' - y_1}{b} = \dfrac{z' - z_1}{c} = -\dfrac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$.

(*iv*) To find image of a point w.r.t. a plane: Let $P(x_1, y_1, z_1)$ is a given point and $ax + by + cz + d = 0$ is given plane. Let (x', y', z') is the image point then $\dfrac{x' - x_1}{a} = \dfrac{y' - y_1}{b} = \dfrac{z' - z_1}{c} = -2\dfrac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$.

10. Angle Between Two Planes: $\cos\theta = \left|\dfrac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}\right|$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$.

The angle θ between the planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by,

$\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$.

Planes are perpendicular if $\vec{n}_1.\vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$ (λ is a scalar.)

11. Angle Bisectors

(*i*) The equations of the planes bisecting the angle between two given planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(*ii*) Bisector of acute/obtuse angle: First make bot the constant terms positive. Then

$$a_1a_2 + b_1b_2 + c_1c_2 > 0 \Rightarrow \text{origin lies on obtuse angle}$$

$$a_1a_2 + b_1b_2 + c_1c_2 < 0 \Rightarrow \text{origin lies in acute angle}$$

12. Family of Planes

(*i*) Any plane through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is $a_1x + b_1y + c_1z + d_1 + \lambda\,(a_2x + b_2y + c_2z + d_2) = 0$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ & $\vec{r}\cdot\vec{n}_2 = d_2$ is $\vec{r}\cdot(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

13. Area of triangle : From two vector $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then area is given by $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$.

14. Volume of a Tetrahedron: Volume of a tetrahedron with vertices $A\,(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and D (x_4, y_4, z_4) is given by

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

A LINE

1. Equation Of A Line

(*i*) A straight line is intersection of two planes.

It is represented by two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

(*ii*) Symmetric form: $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} = r$.

(*iii*) Vector equation: $\vec{r} = \vec{a} + \lambda\vec{b}$.

(*iv*) Reduction of cartesion form of equation of a line to vector form & vice versa

$$\Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow \vec{r} = \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) + \lambda\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$$

2. Angle Between A Plane And A Line:

(*i*) If θ is the angle between line $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$, Then $\sin\theta=\left[\dfrac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{l^2+m^2+n^2}}\right]$.

(*ii*) Vector form: If θ is the angle between a line $\vec{r}=(\vec{a}+\lambda\vec{b})=\vec{r}.\vec{n}=d$ then $\sin\,\theta=\left[\dfrac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right]$.

(*iii*) Condition for perpendicularity $\dfrac{l}{a}=\dfrac{m}{b}=\dfrac{n}{c}$, $\vec{b}\times\vec{n}=0$.

(*iv*) Conditon for parallel $al+bm+cn=0$, $\vec{b}.\vec{n}=0$.

3. Condition For A Line To Lie In A Plane

(*i*) **Cartesian form:** $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ would lie in a plane $ax+by+cz+d=0$, if $ax_1+by_1+cz_1+d=0$ & $al+bm+cn=0$.

(*ii*) **Vector form:** $\vec{r}=\vec{a}+\lambda\vec{b}$ would line in the plane $\vec{r}.\vec{n}=d$ if $\vec{b}.\vec{n}=0$ & $\vec{a}.\vec{n}=d$.

4. Skew Lines:

(*i*) The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines. If $\begin{vmatrix}\alpha'-\alpha & \beta'-\beta & \gamma'-\gamma\\ l & m & n\\ l' & m' & n'\end{vmatrix}\neq 0$. Then lines are skew.

(*ii*) **Vector Form:** For lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ to be skew $(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)\neq 0$.

(*iii*) Shortest distance between line $\vec{r}=\vec{a}_1+\lambda\vec{b}$ & $\vec{r}=\vec{a}_2+\mu\vec{b}$ is $d=\left|\dfrac{(\vec{a}_2-\vec{a}_1)\times\vec{b}}{|\vec{b}|}\right|$.

5. Sphere: General equation of a sphere is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

$(-u,-v,-w)$ is the centre and $\sqrt{u^2+v^2+w^2-d}$ is the radius of the sphere.

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CHAPTER 29

Probability

MUTUALLY EXCLUSIVE EVENTS

A set of events is said to be mutually exclusive if occurrence of one of them precludes the occurrence of any of the remaining events.

Thus, $E_1, E_2, ..., E_n$ are mutually exclusive if and only if $E_i \cap E_j = \phi$ for $i \neq j$.

INDEPENDENT EVENTS

Two events are said to be independent, if the occurrence of one does not depend on the occurrence of the other.

For example, when a coin is tossed twice, the event of occurrence of head in the first throw and the event of occurrence of head in the second throw are independent events.

COMPLEMENT OF AN EVENT

The complement of an event E, denoted by $\overline{E}$ or E' or E^c, is the set of all sample points of the space other than the sample points in E.

For example, when a die is thrown, sample space

$$S = \{1, 2, 3, 4, 5, 6\}.$$

If $E = \{1, 2, 3, 4\}$, then $\overline{E} = \{5, 6\}$.

Note that $E \cup \overline{E} = S$.

MUTUALLY EXCLUSIVE AND EXHAUSTIVE EVENTS

A set of events $E_1, E_2, ..., E_n$ of a sample space S form a mutually exclusive and exhaustive system of events, if (*i*) $E_i \cap E_j = \phi$ for $i \neq j$ and

(*ii*) $E_1 \cup E_2 \cup \ldots \cup E_n = S$

NOTES

(*i*) $O \leq P(E) \leq 1$, i.e. the probability of occurrence of an event is a number lying between 0 and 1.

(*ii*) $P(\phi) = 0$, i.e. probability of occurrence of an impossible event is 0.

(*iii*) $P(S) = 1$, i.e. probability of occurrence of a sure event is 1.

ODDS IN FAVOUR OF AN EVENT AND ODDS AGAINST AN EVENT

If the number of ways in which an event can occur be m and the number of ways in which it does does not occur be n, then

(*i*) odds in favour of the event $= \dfrac{m}{n}$ and

(*ii*) odds against the event $= \dfrac{n}{m}$.

SOME IMPORTANT RESULTS ON PROBABILITY

1. $P(\overline{A}) = 1 - P(A)$.
2. If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cup B)$.
3. If A and B are mutually exclusive events, then $A \cap B = \phi$ and hence $P(A \cap B) = 0$.

   $\therefore \quad P(A \cup B) = P(A) + P(B)$.
4. If A, B, C are any three events, then $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.
5. If A, B, C are mutually exclusive events, then $A \cap B = \phi$, $B \cap C = \phi$, $C \cap A = \phi$, $A \cap B \cap C = \phi$ and hence $P(A \cap B) = 0$, $P(B \cap C) = 0$, $P(C \cap A) = 0$, $P(A \cap B \cap C) = 0$.

   $\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C)$.
6. $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$.
7. $P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B)$.

8. $P(A) = P(A \cap B) + P(A \cap \overline{B})$.

9. $P(B) = P(B \cap A) + P(B \cap \overline{A})$.

10. If $A_1, A_2, ..., A_n$ are independent events, then $P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1) \cdot P(A_2), ..., P(A_n)$.

11. If $A_1, A_2, ..., A_n$ are mutually exclusive events, then

$$P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n).$$

12. If $A_1, A_2, ..., A_n$ are exhaustive events, then $P(A_1 \cup A_2 \cup ... \cup A_n) = 1$.

13. If $A_1, A_2, ..., A_n$ are mutually exclusive and exhaustive events, then

$$P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n) = 1.$$

14. If $A_1, A_2, ..., A_n$ are n events, then

(*i*) $P(A_1 \cup A_2 \cup ... \cup A_n) \leq P(A_1) + P(A_2) + ... + P(A_n)$.

(*ii*) $P(A_1 \cap A_2 \cap ... \cap A_n) \geq 1 - P(\overline{A}_1) - P(\overline{A}_2) ... - P(\overline{A}_n)$.

CONDITIONAL PROBABILITY

$P\left(B/_A\right)$ = Probability of occurrence of A, given that B has already happened.

$$= \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)}$$

1. **Multiplication theorems on probability**

(*i*) If A and B are two events associated with a random experiment, then

$P(A \cap B) = P(A) \cdot P\left(B/_A\right)$, If $P(A) \neq 0$

or $P(A \cap B) = P(B) \cdot P\left(B/_A\right)$, if $P(B) \neq 0$

(*ii*) **Multiplication theorems for independent events:** If A and B are independent events associated with a random experiment, then $P(A \cap B) = P(A) \cdot P(B)$ i.e. the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities. By multiplication theorem, we have $P(A \cap B) = P(A) \cdot P(B/A)$. Since A and B are independent events, therefore

$P(B/A) = P(B)$. Hence, $P(A \cap B) = P(A) \cdot P(B)$.

2. **Probability of at least one of the n independent events:** If $p_1, p_2, p_3 ... p_n$ be the probabilities of happening of n independent events $A_1, A_2, A_3, ... A_n$ respectively, then

(i) Probability of happening none of them = $P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \ldots \cap \overline{A}_n) =$
$P(\overline{A}_1) \cdot P(\overline{A}_2) \cdot P(\overline{A}_3) \ldots P(\overline{A}_n) = (1-p_1)(1-p_2)(1-p_3) \ldots (1-p_n)$.

(ii) Probability of happening at least one of them

$$= P(A_1 \cup A_2 \cup A_3 \ldots \cup A_n) = 1 - P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3) \ldots P(\overline{A}_n)$$
$$= 1 - (1-p_1)(1-p_2)(1-p_3) \ldots (1-p_n)$$

LAW OF TOTAL PROBABILITY

$$P(A) = P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) + \ldots + P(E_n) \cdot P\left(\frac{A}{E_n}\right)$$

Baye's rule as $P(E_i/A) = \dfrac{P(E_i) \cdot P(A/E_i)}{\sum_{k=1}^{n} P(E_k)\, P(A/E_k)}$.

BINOMIAL DISTRIBUTION

The mean, the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.

RANDOM VARIABLE

The expectation (mean) of the random variable X is defined as $E(X) = \sum_{i=1}^{n} p_i x_i$ and the variance of X is defined as

$$\text{var}(X) = \sum_{i=1}^{n} p_i (x_i - E(X))^2 = \sum_{i=1}^{n} p_i x_i^2 - (E(X))^2 .$$

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